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Supplementary Remarks on Frobeniusean Algebras II

By Tadasi NAKAYAMA (Nagoya) and Masatosi IKEDA

An algebra A over a field F is quasi-Frobeniusean[1] if and only if the dualities

$$\alpha_1) \quad l(r(\mathfrak{l})) = \mathfrak{l}, \qquad \alpha_2) \quad r(l(\mathfrak{r})) = \mathfrak{r}$$

between left and right ideals with respect to annihilation hold, where $\mathfrak{l}$ and $\mathfrak{r}$ are left and right ideals respectively, and $l(*)$ and $r(*)$ denote left and right annihilators in A respectively. Further A is Frobeniusean if and only if besides the annihilation dualities $\alpha_1)$ and $\alpha_2)$ also the rank relations

$$\beta_1) \ (\mathfrak{l} : F) + (r(\mathfrak{l}) : F) = (A : F), \quad \beta_2) \ (\mathfrak{r} : F) + (l(\mathfrak{r}) : F) = (A : F)$$

are valid.[2]

The notion of Frobeniusean and quasi-Frobeniusean algebras as well as these duality criteria have been extended to general rings with minimum condition. In this note we shall study some properties of a ring with the duality $\alpha_1)$ for left ideals, and show, among others, that only the duality relations $\alpha_1)$ and $\beta_1)$ or $\alpha_1)$ are sufficient for an algebra to be Frobeniusean or quasi-Frobeniusean respectively.

Let A be a ring with minimum condition for left and right ideals. (We shall understand by a ring always such a ring.) Let N be the radical of A, $A/N = \bar{A} = \bar{A}_1 + \bar{A}_2 + \dots + \bar{A}_k$ be the direct decomposition of $\bar{A}$ into simple two-sided ideals $\bar{A}_\kappa$ and let $f_{(\kappa)}$, $e_{\kappa,i}$, $e_\kappa = e_{\kappa,1}$, $c_{\kappa,ij}$ and $E_\kappa = \sum_{i=1}^{f(\kappa)} e_{\kappa,i}$ $(\kappa = 1, \dots, k)$ have the same meaning as in S. I or Fr. I §1; namely $e_{\kappa,i}$ $(\kappa = 1, \dots, k;\ i = 1, \dots, f_{(\kappa)})$ are mutually

1) See Part I, Proc. Jap. Acad. (1949) (referred to SI) or Nakayama, On Frobeniusean algebras I, II, III, Ann. Math. 40 (1939), 42 (1941), Jap. J. Math 18 (1942) (referred to Fr I, II, III)

2) Of course $\alpha_1)$, $\alpha_2)$ follow respectively from $\beta_1)$, $\beta_2)$, since $l(r(\mathfrak{l})) \supseteqq \mathfrak{l}$, $r(l(\mathfrak{r})) \supseteqq \mathfrak{r}$, $r(l(r(\mathfrak{l}))) = r(\mathfrak{l})$ and $l(r(l(\mathfrak{r}))) = l(\mathfrak{r})$ always. The same is the case with the modified rank relation $\beta')$ in Fr II, Theorem 7; namely $\beta')$ implies $\alpha_1)$ and $\alpha_2)$, and is sufficient, by itself, to secure that a ring A is Frobeniusean, provided A has composition series for left and right ideals.

$$C_l(\mathfrak{l}) = C_r(A/r(\mathfrak{l})) \quad C_r(\mathfrak{r}) = C_l(A/l(\mathfrak{r}))$$

with C_l and C_r denoting left and right composition lengths, are sufficient (and necessary) in order that an algebra or a ring be quasi-Frobeniusean.