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On the Quotient Semi-Group of a Noncommutative Semi-Group

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In this short note we remark, following K. Asano[1]), that a non-commutative semi-group $\mathfrak{g}$ with a certain condition can be embedded into the quotient semi-group G. The necessary and sufficient condition for the existence of the quotient semi-group is the same as the case of a ring. Moreover if $\mathfrak{g}$ is a ring, we can define the addition in G in a natural manner and G is just the quotient ring of $\mathfrak{g}$.

Definition 1. An element λ in a semi-group $\mathfrak{g}$ is called *regular*, if the following two conditions are satisfied: 1) $a\lambda = b\lambda$ $(a, b \in \mathfrak{g})$ implies $a=b$ and 2) $\lambda a = \lambda b$ $(a, b \in \mathfrak{g})$ implies $a=b$.

If $\mathfrak{g}$ has the unit, the elements having their inverse elements in $\mathfrak{g}$ are obviously regular.

In the following we assume that a semi-group $\mathfrak{g}$ has regular elements. It is clear that all regular elements in $\mathfrak{g}$ form a sub-semi-group $\mathfrak{g}^*$ of $\mathfrak{g}$.

Definition 2. Let $\mathfrak{m}$ be a sub-semi-group of $\mathfrak{g}^*$. If a semi-group G which contains $\mathfrak{g}$ satisfies the next three conditions, we call G a *left quotient semi-group* of $\mathfrak{g}$ by $\mathfrak{m}$.

(1) G has a unit 1.

(2) Every element α in $\mathfrak{m}$ has an inverse α^{-1} in G.

(3) For every x in G, there exists α in $\mathfrak{m}$ such that αx is contained in $\mathfrak{g}$.

In particular if $\mathfrak{m} = \mathfrak{g}^*$, we call G a left quotient semi-group of $\mathfrak{g}$. According to Definition 2, every element s in G is clearly expressible in the form $s = \alpha^{-1}a$, where $\alpha \in \mathfrak{m}$ and $a \in \mathfrak{g}$. If $\mathfrak{g}$ has a left (or right) unit e, then $e = 1$.[2])

Lemma 1. If for every a in $\mathfrak{g}$ and every α in $\mathfrak{m}$ there exist α' in $\mathfrak{m}$ and a' in $\mathfrak{g}$ such that $\alpha' a = a' \alpha$ then, for any n elements $\lambda_i \in \mathfrak{m}$ $(i=1, \dots, n)$ there exist n elements $c_i \in \mathfrak{g}$ $(i=1, \dots, n)$ satisfying the following condition:

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1) K. Asano, Arithmetische Idealtheorie in nichtkommutativen Ringen, Japan. Journ. Math. **16** (1939); Über die Quotientenbildung von Schiefringen, Journ. Math. Soc. Japan **1** (1949).

2) $e = e1 = e\lambda\lambda^{-1} = \lambda\lambda^{-1} = 1$ $(e = \lambda e = \lambda^{-1}\lambda e = \lambda^{-1}\lambda = 1)$.