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On some types of convergence of positive definite functions

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1. INTRODUCTION. Let us denote by G an arbitrary but fixed locally compact group and by $m(\cdot)$ a left-invariant Haar measure on G. A complex-valued function $\varphi(g)$ on G is called positive definite if it satisfies

( ) $$\sum_{j,k} \bar{\xi}_k \xi_j \varphi(g_k^{-1} g_j) \geqq 0 \text{ [1]},$$

for complex $\xi_k (1 \leqq k \leqq n)$. We shall denote by $\boldsymbol{P}$ the set of all continuous [2]) positive definite functions. Then we can define in $\boldsymbol{P}$, among others, the following two types of topology:

(P) *Pontrjagin's topology*: the neighbourhood system of φ_0 of $\boldsymbol{P}$ consists of all sets of the following form:

$$\{ \varphi \mid |\varphi(g) - \varphi_0(g)| < \varepsilon \text{ on } F\},$$

where $\varepsilon > 0$ and F is a compact set in G.

(W) *Weak topology*: Let $L^1 \equiv L^2(G)$ be the Banach space of all complex-valued m-integrable functions on G. Then to every φ of $\boldsymbol{P}$ there corresponds a functional on L^1 if we define

$$(\varphi, x) = \int x(g) \varphi(g) \, dg \text{ [3]},$$

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1) $\bar{z}$ denotes the conjugate complex number of z.

2) We can also define the positive definiteness of a bounded measurable function φ as follows:

(**) $$\int \int \varphi(h^{-1}g) \overline{x(h)} x(g) \, dh \, dg \geqq 0 \quad \text{for} \quad x(\cdot) \text{ of } L^1(G).$$

Every measurable function satisfying (*) satisfies (**), too. On the other hand it is proved that every (**)-positive definite function coincides almost everywhere with a continuous one. (Cf. the immediately preceeding note in this Journal by the present author. We can also prove this continuity by using the arguments in the present paper and refering some of those in that paper, without making explicit use of the unitary representation of G, though, in essential, both proofs depend on the same idea. Cf. also [1] in LITERATURE at the end of this paper.)

3) The integration is relative to $m(\cdot)$ and its domain is G.