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GENUS ONE 1-BRIDGE KNOTS AS VIEWED FROM THE CURVE COMPLEX

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1. Introduction

W.J. Harvey [4] associated to a surface S a finite-dimensional simplicial complex $C(S)$, called the curve complex, which we recall below.

For a connected orientable surface $F = F_{g,n}$ of genus g with n punctures, the *curve complex* $C(F)$ of F is the complex whose k-simplexes are the isotopy classes of $k+1$ collections of mutually non-isotopic essential loops in F which can be realized disjointly. It is proved in [16] that the curve complex is connected if F is not sporadic (where F is *sporadic* if $g = 0$, $n \leq 4$ or $g = 1$, $n \leq 1$). For $[x]$ and $[y]$, vertices of $C(F)$, the *distance* $d([x],[y])$ between $[x]$ and $[y]$ is defined by the minimal number of 1-simplexes in a simplicial path joining $[x]$ to $[y]$. It is known that if S is not sporadic, then $C(F)$ has infinite diameter with respect to the distance defined above (cf. [11], [16]), $C(F)$ is not locally finite in the sense that there are infinite edges around each vertex, and the dimension of $C(F)$ is $3g-4+n$.

Recently, J. Hempel [11] studied Heegaard splittings of closed 3-manifolds by using the curve complex of Heegaard surfaces. Let M be a closed orientable 3-manifold and $(V_1, V_2; S)$ a genus $g \geq 2$ Heegaard splitting, that is, V_i ($i = 1$ and 2) is a genus g handlebody with $M = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 \cap \partial V_2 = S$. By using the curve complex, Hempel defined the distance of the Heegaard splitting, denoted by $d(V_1, V_2)$, and proved the following results.

Theorem 1.1 (J. Hempel). (1) *Let M be a closed, orientable, irreducible 3-manifold which is Seifert fibered or which contains essential tori. Then $d(V_1, V_2) \leq 2$ for any Heegaard splitting $(V_1, V_2; S)$ of M.*
(2) *There are Heegaard splittings of closed orientable 3-manifolds with distance $> n$ for any integer n.*

In particular, the theorem above implies that a Haken manifold is hyperbolic if a Heegaard splitting of the manifold has distance ≥ 3. Results along these lines were also obtained by A. Thompson [20]. Moreover, H. Goda, C. Hayashi and N. Yoshida [2] made detailed study of tunnel number one knots and C. Hayashi ([6], [7]) studied $(1,1)$-knots from similar points of view.