Kohnen, W. and Salvati Manni, R.
Osaka J. Math.
41 (2004), 353–356

LINEAR RELATIONS BETWEEN THETA SERIES

WINFRIED KOHNEN and RICCARDO SALVATI MANNI

(Received June 27, 2003)

1. Introduction

For a positive integer $m \equiv 0 \pmod 8$, let $S_1, \dots, S_{h(m)}$ be a complete set of representatives of $GL_m(\mathbf{Z})$-classes of positive definite, symmetric, even integral, unimodular matrices of size m. If n is a positive integer, then in the usual way one can attach to S_ν ($\nu = 1, \dots, h(m)$) a theta series $\vartheta_{S_\nu}^{(n)}$ which is a modular form of weight $m/2$ and genus n on the Siegel modular group $\Gamma_n := Sp_n(\mathbf{Z}) \subset GL_{2n}(\mathbf{Z})$. For each ν, the sequence $(\vartheta_{S_\nu}^{(n)})_{n\geq 1}$ is a so-called stable system of Siegel modular forms, i.e. $\vartheta_{S_\nu}^{(n)}$ for each $n \geq 1$ is obtained from $\vartheta_{S_\nu}^{(n+1)}$ by applying the Siegel Φ-operator.

It is widely believed that for $m \geq 24$ the series $\vartheta_{S_\nu}^{(n)}$ ($\nu = 1, \dots, h(m)$) are linearly independent if and only if $n \geq m/2$. The latter assertion was proved by Borcherds, Freitag and Weissauer [2] in the first case $m = 24$. Nothing so far seems to be known for $m > 24$. However, one of the authors [6] using a similar method as in [2] proved the partial result that for arbitrary $m \equiv 0 \pmod{24}$ the series $\vartheta_{S_\nu}^{((m/2)-1)}$ ($\nu = 1, \dots, h(m)$) are linearly dependent.

The main purpose of this note is to show that on average as $m \to \infty$, $m \equiv 0 \pmod 8$ there are in fact at least $(1/2)[m/24]$ independent linear relations between the above series. For aprecise statement see the Theorem in Section 2.

The proof—quite different from the ideas of [2, 6]—combines three inputs: firstly Ikeda's lifting theorem [4], secondly the characterization of Hecke eigenforms lying in the space generated by the theta series in terms of the behavior of their standard L-functions at special points due to Böcherer [1] and Weissauer [7], and thirdly a result of Iwaniec and Sarnak [5] on the non-vanishing of central critical values of Hecke L-functions of elliptic cuspidal Hecke eigenforms.

At the end of the paper, in Section 3 we will make some further remarks on the relation between Ikeda's lifting theorem and theta series. We remark that the relevance of the Ikeda lift in connection with theta series was also noticed by Conrey and Farmer in [3].

2. Linear relations

For k an integer and n a positive integer we denote by $M_k(\Gamma_n)$ the space of Siegel modular forms of weight k with respect to Γ_n and by $S_k(\Gamma_n)$ the subspace of