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ON THE CHARACTERIZATION OF CANONICAL NUMBER SYSTEMS

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1. Introduction

It is well known that each positive integer n can be expressed uniquely as a sum $n = d_0 + d_1 b + \cdots + d_h b^h$ with an integral base number $b \geq 2$, $d_h \neq 0$ and $d_i \in \{0, \ldots, b-1\}$. This concept can be generalized in several directions.

On the one hand the base sequence 1, b, b^2, ... can be replaced by a sequence $1 = u_0 < u_1 < u_2 < \cdots$ to obtain representations of positive integers. Of special interest is the case where the sequence $\{u_i\}_{i=0}^{\infty}$ is defined by a linear recurrence. A famous example belonging to this class is the so-called Zeckendorf representation.

On the other hand, one can generalize the set of numbers which can be represented. We mention two kinds of number systems belonging to this class:

The so called β-expansions introduced by Rényi [27] which are representations of real numbers in the unit interval as sums of powers of a real base number β. These digit representations of real numbers are strongly related to digit representations of positive integers if β is a zero of the characteristic polynomial of a linear recurring base sequence $\{u_i\}_{i=0}^{\infty}$. Of special interest is the case where β is a Pisot number. These expansions have been extensively studied. We mention here the papers Berend-Frougny [6] , Frougny [12, 13], Frougny-Solomyak [14, 15] and Loraud [25] and refer to the references given there.

Another kind of number systems which admit the representation of a set which is different from $\mathbb{N}$ are the so-called *canonical number systems* (for short CNS). Since CNS form the main object studied in the present paper we recall their definition (cf. Akiyama-Pethő [2]).

DEFINITION 1.1. Let

$$P(x) := b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0 \in \mathbb{Z}[x]$$

be such that $n \geq 1$ and $b_n = 1$ (set $b_j = 0$ for $j > n$). Let $\mathcal{N} = \{0, 1, \ldots, |b_0| - 1\}$ and

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