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NORMAL AFFINE SURFACES WITH $\mathbb{C}^*$-ACTIONS

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Introduction

A classification of (normal) affine surfaces admitting a $\mathbb{C}^*$-action was given e.g., in [5, 6, 21, 22, 1, 25] and [12]–[14]. Here we obtain a simple alternative description of normal affine surfaces V with a $\mathbb{C}^*$-action in terms of their graded coordinate rings as well as by defining equations. Our approach is based on a generalization of the Dolgachev-Pinkham-Demazure construction [11, 22, 10]. Recall (see [12]–[14]) that a $\mathbb{C}^*$-action on a normal affine surface V is called *elliptic* if it has a unique fixed point which belongs to the closure of every 1-dimensional orbit, *parabolic* if the set of its fixed points is 1-dimensional, and *hyperbolic* if V has only a finite number of fixed points, and these fixed points are of hyperbolic type, that is each one of them belongs to the closure of exactly two 1-dimensional orbits.

In the elliptic case, the complement V^* of the unique fixed point in V is fibered by the 1-dimensional orbits over a projective curve C. In the other two cases V is fibered over an affine curve C, and this fibration is invariant under the $\mathbb{C}^*$-action.

Vice versa, given a smooth curve C and a $\mathbb{Q}$-divisor D on C, the Dolgachev-Pinkham-Demazure construction provides a normal affine surface $V = V_{C,D}$ with a $\mathbb{C}^*$-action such that C is just the algebraic quotient of V^* or of V, respectively. This surface V is of elliptic type if C is projective and of parabolic type if C is affine.

We remind this construction in Sections 1 and 2 below. In Section 3 we use it to present any normal affine surface V with a parabolic $\mathbb{C}^*$-action as a normalization of the surface $x^d - P(z)y = 0$ in $\mathbb{A}^3_{\mathbb{C}}$ for a certain $d \in \mathbb{N}$ and a certain polynomial $P \in \mathbb{C}[t]$ (see Theorem 3.11).

In Section 4 we deal with the hyperbolic case. We generalize the Dolgachev-Pinkham-Demazure construction in order to make it work for any hyperbolic $\mathbb{C}^*$-surface. Instead of one $\mathbb{Q}$-divisor D on a smooth affine curve C as before, it involves now two $\mathbb{Q}$-divisors D_+ and D_- on C. By our result *isomorphism classes of normal affine hyperbolic* $\mathbb{C}^*$*-surfaces are in* 1-1*-correspondence to equivalence classes*

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