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A CRITERION OF EXACTNESS OF THE CLEMENS-SCHMID SEQUENCES ARISING FROM SEMI-STABLE FAMILIES OF OPEN CURVES

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1. Introduction

Let $\pi\colon X \to \Delta$ be a semistable family of projective algebraic varieties over the unit disk. In other words, π is flat and projective over Δ, smooth over $\Delta^* = \Delta - \{0\}$, and the central fiber $Y = \pi^{-1}(0)$ is a divisor with normal crossings without multiple components. We write $X_t := \pi^{-1}(t)$ for $t \in \Delta^*$ and $X^* := X - Y$. There is the Wang exact sequence

$$\cdots \to H^q(X^*, \mathbf{Q}) \to H^q(X_t, \mathbf{Q}) \xrightarrow{N} H^q(X_t, \mathbf{Q}) \to H^{q+1}(X^*, \mathbf{Q}) \to \cdots, \tag{1.1}$$

and the localization exact sequence

$$\cdots \to H^q_Y(X, \mathbf{Q}) \to H^q(X, \mathbf{Q}) \to H^q(X^*, \mathbf{Q}) \to H^{q+1}_Y(X, \mathbf{Q}) \to \cdots. \tag{1.2}$$

Here N denotes the log monodromy around $\Delta^* = \Delta - \{0\}$. Combining those sequences and the natural isomorphism $H^q(X, \mathbf{Q}) \simeq H^q(Y, \mathbf{Q})$, we obtain a sequence

$$\cdots \to H^{q-2}(X_t, \mathbf{Q}) \to H^q_Y(X, \mathbf{Q}) \to H^q(Y, \mathbf{Q}) \to H^q(X_t, \mathbf{Q}) \xrightarrow{N} H^q(X_t, \mathbf{Q}) \to \cdots. \tag{1.3}$$

This is called the *Clemens-Schmid sequence*. A theorem of Clemens and Schmid says that the sequence (1.3) is exact ([1, §3]). In particular, the first piece

$$H^q(Y, \mathbf{Q}) \longrightarrow H^q(X_t, \mathbf{Q}) \xrightarrow{N} H^q(X_t, \mathbf{Q}) \tag{1.4}$$

is called the *local invariant cycle theorem* ([3, (5.12)]).

In [2, (12.3.1)], S. Usui et al. proposed a problem whether the Clemens-Schmid sequence (1.3) or (1.4) is exact when we remove the assumption that π is proper. In this paper, we give a necessary and sufficient condition for that the sequence (1.4) is exact when π is a semistable family of open curves. In particular, we see that the Clemens-Schmid sequences of non-proper families are not exact in general.

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