Sugita, H. and Takanobu, S.
Osaka J. Math.
40 (2003), 945–976

THE PROBABILITY OF TWO INTEGERS TO BE CO-PRIME, REVISITED — ON THE BEHAVIOR OF CLT-SCALING LIMIT

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(Received February 27, 2002)

1. Introduction

Let $\gcd(x, y)$ denote the greatest common divisor of integers x and y. Define functions X and S_N on $\mathbb{Z}^2$ by

$$X(x, y) := \begin{cases} 1, & \text{if } \gcd(x, y) = 1, \\ 0, & \text{if } \gcd(x, y) > 1, \end{cases} \tag{1}$$

$$S_N(x, y) := \frac{1}{N^2} \sum_{m,m'=1}^{N} X(x + m, y + m'), \quad N \in \mathbb{N}. \tag{2}$$

The following number-theoretic limit theorem is due to Dirichlet [4] (cf. [6, Theorem 332]):

$$\lim_{N\to\infty} S_N(x, y) = \frac{6}{\pi^2}, \quad (x, y) \in \mathbb{Z}^2. \tag{3}$$

Regarding (3) as a law of large numbers (LLN for short), it is natural to ask if a central limit theorem (CLT for short) holds for X. That is, for sufficiently large N, is the scaled function

$$Y_N(x, y) := N \left(S_N(x, y) - \frac{6}{\pi^2} \right) \tag{4}$$

approximately normally "distributed"? Here we consider "distribution" of Y_N, at the suggestion of [2], [5] and [9], as follows: If the limit

$$\lim_{M\to\infty} \frac{1}{M^2} \sum_{m,m'=1}^{M} \exp\left(\sqrt{-1}\, t\, Y_N(m, m')\right), \quad t \in \mathbb{R}, \tag{5}$$

2000 *Mathematics Subject Classification* : Primary 60B10; secondary 60F05, 60B15, 11N37, 11K41.

*Partially supported by Grant-in-Aid for scientific research 11440034 Min. Education, Japan.
†Partially supported by Grant-in-Aid for scientific research 13640108 Min. Education, Japan.