Kokubu, M., Umehara, M. and Yamada, K.
Osaka J. Math.
40 (2003), 697–715

AN ELEMENTARY PROOF OF SMALL'S FORMULA FOR NULL CURVES IN PSL(2, *C*) AND AN ANALOGUE FOR LEGENDRIAN CURVES IN PSL(2, *C*)

Dedicated to Professor Katsuei Kenmotsu on his sixtieth birthday

MASATOSHI KOKUBU, MASAAKI UMEHARA and KOTARO YAMADA

(Received January 17, 2002)

1. Introduction

Let M^2 be a Riemann surface, *which might not be simply connected*. A meromorphic map F from M^2 into $\mathrm{PSL}(2,\boldsymbol{C}) = \mathrm{SL}(2,\boldsymbol{C})/\{\pm\,\mathrm{id}\}$ is a map which is represented as

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \sqrt{h}\begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} \qquad \bigl(AD - BC = 1\bigr), \tag{1.1}$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ and h are meromorphic functions on M^2. Though $\sqrt{h}$ is a multi-valued function on M^2, F is well-defined as a $\mathrm{PSL}(2,\boldsymbol{C})$-valued mapping.

A meromorphic map F as in (1.1) is called a *null curve* if the pull-back of the Killing form by F vanishes, which is equivalent to the condition that the derivative $F_z = \partial F/\partial z$ with respect to each complex coordinate z is a degenerate matrix everywhere. It is well-known that the projection of a null curve in $\mathrm{PSL}(2,\boldsymbol{C})$ into the hyperbolic 3-space $H^3 = \mathrm{PSL}(2,\boldsymbol{C})/\,\mathrm{PSU}(2)$ gives a constant mean curvature one surface (see [2, 10]). For a non-constant null curve F, we define two meromorphic functions

$$G := \frac{dA}{dC} = \frac{dB}{dD}, \qquad g := -\frac{dB}{dA} = -\frac{dD}{dC}. \tag{1.2}$$

(For a precise definition, see Definition 2.1 in Section 2). We call G the *hyperbolic Gauss map* of F and g the *secondary Gauss map*, respectively [12]. In 1993, Small [8] discovered the following expression

$$F = \begin{pmatrix} G\dfrac{da}{dG} - a & G\dfrac{db}{dG} - b \\ \dfrac{da}{dG} & \dfrac{db}{dG} \end{pmatrix}, \quad \left(a := \sqrt{\frac{dG}{dg}},\ b := -ga\right) \tag{1.3}$$

2000 *Mathematics Subject Classification* : Primary 53A10; Secondary 53A35, 53A07.