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SOME PROPERTIES OF INVERTIBLE SUBSTITUTIONS OF rank d, AND HIGHER DIMENSIONAL SUBSTITUTIONS

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0. Introduction

We denote by $\mathcal{A}_d{}^*$ (resp., F_d) the free monoid (resp., the free group), with the empty word as unit, generated by an alphabet $\mathcal{A}_d := \{1, 2, \dots, d\}$ consisting of d letters. We consider an endomorphism σ on F_d, i.e., a group homomorphism from F_d to itself. An endomorphism σ will be referred to as a *substitution* if we can take a nonempty word $\sigma(i) \in \mathcal{A}_d{}^*$ for all $i \in \mathcal{A}_d$, cf. the first paragraph of Section 1. When is a substitution σ invertible as an endomorphism on F_d? An answer to this question is known when $d = 2$, cf. Proposition 1. Our objective is to generalize Proposition 1 for arbitrary $d \geq 2$. We introduce a geometrical method in [2]; and we use a general method given in [6], where the so called *higher dimensional substitutions* $E_k(\sigma)$ $(0 \leq k \leq d)$ are established for a given substitution σ on F_d.

Throughout the paper, we denote by $\boldsymbol{Z}$ (resp., $\boldsymbol{N}$, $\boldsymbol{R}$) the set of integers (resp., positive integers, real numbers), and by $\mathrm{End}(F_d)$ (resp., $\mathrm{Sub}(F_d)$, $\mathrm{Aut}(F_d)$, $\mathrm{IS}(F_d)$) the set of endomorphisms (resp., substitutions, automorphisms, invertible substitutions) on F_d.

Let $d \geq 2$ be an integer. We mean by $(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k)$ the positively oriented unit cube of dimension k translated by $\boldsymbol{x}$ in the Euclidean space $\boldsymbol{R}^d$:

$$(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k) := \{\boldsymbol{x} + t_1\boldsymbol{e}_{i_1} + \cdots + t_k\boldsymbol{e}_{i_k} \mid 0 \leq t_n \leq 1,\ 1 \leq n \leq k\},$$
$$\boldsymbol{x} \in \boldsymbol{Z}^d,\ 0 \leq k \leq d,\ 1 \leq i_1 < \cdots < i_k \leq d,$$

where $\{\boldsymbol{e}_i\}_{i=1,\dots,d}$ is the canonical basis of $\boldsymbol{R}^d$. In particular, for $k = 0$, the k dimensional unit cube $(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k)$, which will be denoted by $(\boldsymbol{x}, \bullet)$, is considered to turn out a point $\boldsymbol{x}$. In general, for $\{i_1, i_2, \dots, i_k\}$ with $1 \leq i_m \leq d$, $1 \leq m \leq k$, we define

$$(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k) := 0,\ \text{ if } i_n = i_m \text{ for some } n \neq m,$$
$$(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k) := \epsilon(\tau)(\boldsymbol{x}, i_{\tau(1)} \wedge \cdots \wedge i_{\tau(k)}) \quad (1 \leq i_{\tau(1)} < \cdots < i_{\tau(k)} \leq d), \quad \text{otherwise},$$

where τ is a permutation on $\{1, \dots, k\}$, and $\epsilon(\tau)$ is the signature of τ, which designates the orientation. We put

$$\Lambda_0 := \boldsymbol{Z}^d \times \{\bullet\},$$