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A MÖBIUS INVERSION FORMULA FOR GENERALIZED LEFSCHETZ NUMBERS

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1. Introduction

Let G be a finite group and $f : U \to X$ an equivariant map. A common way of studying the properties of f is looking at the restrictions $f^H : X^H \to Y^H$ to the spaces fixed by the subgroups H of G, as non-equivariant maps. For example, if U and X are G-CW complexes, then $f : U \to X$ is a G-equivariant homotopy equivalence if and only if for every H the map f^H is a homotopy equivalence; a similar result related to a G-retraction is due to Jaworowski: a locally compact, separable metric and finite-dimensional G-space X is a G-ENR if and only if for every H the fixed point set X^H is an ENR [11]. This paper is addressed to studying fixed points (up to compactly fixed G-homotopy) of a G-equivariant self-map $f : U \subset M \to M$, where M is a G-ENR or a smooth G-manifold. If there is a compactly fixed G-homotopy f_t, $t \in I$, such that $f_0 = f$ and f_1 is fixed point free, then for every subgroup H there is a compactly fixed homotopy f_t^H such that $f_0^H = f^H$ and f_1^H is fixed point free, and this means that every restriction f^H can be deformed to a fixed point free map. To investigate under which conditions the converse of this statement is true, it is necessary to exhibit the algebraic obstructions of the existence of the equivariant deformation f_t, and then relate them to the corresponding obstructions of the non-equivariant restrictions f^H. Under some dimensional assumptions, Nielsen theory is exactly what describes these invariants; if M^H is a manifold of dimension different from 2 then the generalized Lefschetz number $\mathcal{L}(f^H)$ or equivalently the Nielsen number $N(f^H)$ vanish if and only if f^H can be deformed to be fixed point free (it is the Converse of the Lefschetz Property). So the problem can be stated algebraically as: under which conditions does the knowledge of the generalized Lefschetz numbers $\mathcal{L}(f^H)$ allow to compute the obstructions to an equivariant deformation? Again, it is necessary to relate the latter obstruction to a set of invariants (as done first in [7]), namely the generalized Lefschetz numbers of some restrictions $\mathcal{L}({f'}^H|U_H)$ of a suitable approximation f' of f. This set of homotopy invariants gives what might be thought as an equivariant generalized Lefschetz number, and, under the same dimension assumptions as above, they vanish if and only if the map has a G-deformation to a fixed point free map (compactly fixed). So the point is to relate the two sets of invariants above.

First results in this direction date back to the papers of Komiya [13],