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CONJUGATE CONNECTIONS AND $SU(3)$-INSTANTON INVARIANTS

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1. Introduction

In their paper [8], S. Kobayashi and E. Shinozaki introduced the concept of conjugate connection for a reducible principal bundle P. From the point of view of the geometry of 4-manifolds, the importance of this concept is that an automorphism of a Lie group G fixing a Lie subgroup H induces a compatible action on the quotient space $\mathcal{B}(P)$ of connections by the group of gauge transformations. When we fix a Riemannian metric g on the base manifold and G is compact and semi-simple, we can also see that the automorphism group acts on the moduli space $\mathcal{M}_P(g)$ of Yang-Mills connections modulo the group of gauge transformations. In particular, the automorphism group acts on the moduli space of anti-self-dual (ASD) connections, when the dimension of the base manifold is 4.

One advantage of this action on the moduli space of ASD connections is that it is not the action induced from an action on the base manifold. When we attempt to use the induced action on the moduli space coming from the base manifold, typically we have to overcome some serious transversality issues. M. Itoh, however, observed in [5] that the inner automorphism group induces a trivial action on the quotient space $\mathcal{B}(P)$. It is well known [12] that for compact simple Lie groups only Lie groups of type A_r ($r > 1$), D_r ($r \geq 4$), and E_6 have a nontrivial outer automorphism group. As a first application of conjugate connections to the geometry of 4-manifolds, in this paper we exclusively deal with the simplest compact simple Lie group $SU(3)$ other than $SU(2)$. In this case the outer automorphism group is isomorphic to the cyclic group $\mathbb{Z}_2$.

One of the aims of this paper is to prove a fixed point theorem under the group $\mathbb{Z}_2$ action on the irreducible part $\mathcal{B}^*(P)$ of the quotient space $\mathcal{B}(P)$ in a reducible principal $SU(3)$-bundle P along the irreducible part $\mathcal{B}^*(Q)$ of the quotient spaces $\mathcal{B}(Q)$ of $SO(3)$-subbundles Q of P. More precisely, in Section 3 we show the following

Theorem 1.1. *Let X be a closed oriented simply-connected manifold. Fix a Riemannian metric g on X, and let P be a principal $SU(3)$-bundle reducible to an $SO(3)$-subbundle Q. Let $\mathcal{M}^*_P(g)_\sigma$ denote the fixed point set of the irreducible part $\mathcal{M}^*_P(g)$ of the moduli space $\mathcal{M}_P(g)$ of Yang-Mills (or ASD) on P under the action of the outer automorphism group of $SU(3)$ fixing a Lie subgroup $SO(3)$ generated by σ.*