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# WEAKLY HYPERBOLIC EQUATIONS, SOBOLEV SPACES OF VARIABLE ORDER, AND PROPAGATION OF SINGULARITIES

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## 1. Introduction

Let us recall some results about linear and semilinear wave equations. We examine the Cauchy problems

$$\Box u = f(u), \qquad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.1}$$

$$\Box v = 0, \qquad v(0,x) = u_0(x), \quad v_t(0,x) = u_1(x), \tag{1.2}$$

where $\Box = \partial_t^2 - \triangle$, $(t,x) \in \mathbb{R}_t \times \mathbb{R}_x^n$ and $f \in C^\infty$ with $f(0) = 0$. We suppose that the initial data $u_0$, $u_1$ satisfy $u_0 \in H^s$, $u_1 \in H^{s-1}$ for some $s > n/2 + 1$. Then it is known that solutions $u, v$ exist in $C([0,T], H^s) \cap C^1\left([0,T], H^{s-1}\right)$ for some small $T > 0$. Further, we assume that $u_0$, $u_1$ belong to $C^\infty$ outside some closed set of $\mathbb{R}^n$. If singularities starting from two different points of this set of singularities meet, nothing happens in the linear case. They ignore each other and continue on their track. However, in the semilinear case, the nonlinear interaction of singularities may generate new singularities. These are weaker than those of $v$ by at least one Sobolev order, which can be seen immediately as follows: we have $\Box(u - v) = f(u) \in C([0,T], H^s)$; hence $u - v \in C\left([0,T], H^{s+1}\right)$.

The aim of this publication is to prove a similar result for weakly hyperbolic equations whose lower order terms satisfy sharp Levi conditions. To demonstrate the phenomena which may occur in this setting, we recall a result of [15]. Let $v = v(t,x)$ be the solution of

$$v_{tt} - t^2 v_{xx} - b v_x = 0, \quad v(0,x) = u_0(x), \quad v_t(0,x) = 0, \quad x \in \mathbb{R}. \tag{1.3}$$

If $b = 4m + 1$ and $m \in \mathbb{N}_0$, then the solution is given by

$$v(t,x) = \sum_{j=0}^{m} C_{jm} t^{2j} (\partial_x^j u_0)\left(x + \frac{t^2}{2}\right) \tag{1.4}$$

with some constants $C_{jm}$; and $C_{mm}$ does not vanish. We observe two phenomena. The first is the *loss of regularity*: if $u_0 \in H^s$, then $v(t,.) \in H^{s-m}$. There is *no classical*