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RANK RIGIDITY, CONES, AND CURVATURE-HOMOGENEOUS HADAMARD MANIFOLDS

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1. Introduction

The rank of a geodesic in a Riemannian manifold is the dimension of the real vector space of all parallel Jacobi vector fields along it. The rank of the manifold is the minimum of the ranks of all its geodesics. A manifold is said to have higher rank if its rank is greater than one. Higher rank seems to be rather exceptional. For instance, any locally irreducible, compact, Riemannian manifold with nonpositive sectional curvature and higher rank is a locally symmetric space ([1], [3], [4]). The curvature assumption is essential ([8]). It is an open problem whether the compactness assumption can be weakened to completeness (*Rank Rigidity Conjecture*, [2]).

In the present paper we approach this problem via infinitesimal geometry. We give examples of connected, simply connected, locally irreducible Riemannian manifolds with nonpositive sectional curvature and higher rank but which are not locally symmetric. These examples are not complete, but they show that the local Rank Rigidity Conjecture does not hold and that rank rigidity is a global phenomenon. We also study Riemannian manifolds which are almost flat in some infinitesimal sense. Among them are some irreducible, curvature-homogeneous, inhomogeneous, Hadamard manifolds. In view of rank rigidity for homogeneous Hadamard manifolds [6] it is natural to investigate the rank of these manifolds. We show that these manifolds have indeed rank one.

We now describe the contents of this paper in more detail. The *infinitesimal rank* of a Riemannian manifold M is the largest integer k such that for every $p \in M$ and $v \in T_pM$ there exists a k-dimensional subspace F (*infinitesimal k-flat*) of T_pM with $v \in F$ and $R(X,Y)Z = 0$ for all $X, Y, Z \in F$, where R is the curvature tensor of M. The n-dimensional Riemannian manifolds with infinitesimal rank n are obviously the flat Riemannian manifolds. In Section 2 we classify the n-dimensional Riemannian manifolds with infinitesimal rank $n-1$. Since the null space of the Jacobi operator of these manifolds has codimension one these manifolds are good candidates for having many parallel Jacobi vector fields. Among them are certain cones and certain Hadamard manifolds which can be realized as twisted products. In Section 3 we present some formulas for the curvature of twisted products. In Section 4 we show that

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