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HARMONIC COHOMOLOGY GROUPS ON COMPACT SYMPLECTIC NILMANIFOLDS

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1. Introduction

Let (M^{2m}, ω) be a symplectic manifold. Brylinski [2] defined the star operator $*\colon \Omega^k(M) \to \Omega^{2m-k}(M)$ for the symplectic structure ω as an analogy of the star operator for an oriented Riemannian manifold, where $\Omega^k(M)$ denotes the space of all k-forms on M, and also defined an operator $d^* = (-1)^k * d *\colon \Omega^k(M) \to \Omega^{k-1}(M)$. Now a form α on M is called a symplectic harmonic form if it satisfies $d\alpha = d^*\alpha = 0$. We denote by $\mathcal{H}^k(M)$ the space of all harmonic k-forms on M. We define symplectic harmonic k-cohomology group $H^k_{hr}(M)$ by $\mathcal{H}^k(M)/(B^k(M) \cap \mathcal{H}^k(M))$. Brylinski conjectured that any de Rham cohomology class contains a harmonic representation. However, Mathieu [6] proved the following result:

Mathieu's Theorem. *Let (M^{2m}, ω) be a symplectic manifold of dimension $2m$. Then following two assertions are equivalent*:
(a) *For any k, the cup-product $[\omega]^k\colon H^{m-k}_{DR}(M) \to H^{m+k}_{DR}(M)$ is surjective.*
(b) *For any k, $H^k_{DR}(M) = H^k_{hr}(M)$.*

In particular, we see that if M is a compact Kähler manifold, then any de Rham cohomology class contains a symplectic harmonic cocycle. Yan [11] gave a simpler, more direct proof of Mathieu's Theorem. Mathieu [6] also proved that, for $k = 0, 1, 2$, $H^k_{DR}(M) = H^k_{hr}(M)$.

In this paper we study compact symplectic nilmanifolds. Let $\mathfrak{g}$ be a Lie algebra and put $\mathfrak{g}^{(0)} = \mathfrak{g}$ and let $\mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}]$ for $i \geq 0$. We say that a Lie algebra $\mathfrak{g}$ is $(r+1)$-step nilpotent if $\mathfrak{g}^{(r)} \neq (0)$ and $\mathfrak{g}^{(r+1)} = (0)$. A Lie group G is called $(r+1)$-step nilpotent if its Lie algebra $\mathfrak{g}$ is $(r+1)$-step nilpotent. If G is a simply-connected $(r+1)$-step nilpotent Lie group and Γ is a lattice of G, that is, a discrete subgroup of G such that G/Γ is compact, then we say that G/Γ is a compact $(r+1)$-step nilmanifold. We also identify $\bigwedge \mathfrak{g}^*$ with the space of all left G-invariant forms on G. Nomizu [8] proved that, for each k, the Lie algebra cohomology group $H^k(\mathfrak{g}) = Z^k(\mathfrak{g})/B^k(\mathfrak{g}) = (\operatorname{Ker} d \cap \bigwedge^k(\mathfrak{g}^*))/(\operatorname{Im} d \cap \bigwedge^k(\mathfrak{g}^*))$ is isomorphic to the de Rham cohomology group $H^k_{DR}(M) = Z^k(M)/B^k(M) = (\operatorname{Ker} d \cap \Omega^k(M))/(\operatorname{Im} d \cap \Omega^k(M))$, where $M = G/\Gamma$.