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EINSTEIN METRICS ON BOGGINO-DAMEK-RICCI TYPE SOLVABLE LIE GROUPS

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1. Introduction

A Riemannian manifold (M, g) is called an Einstein manifold if its Ricci tensor ric_g satisfies $\mathrm{ric}_g = cg$ for some constant c. This paper deals with noncompact homogeneous Einstein manifolds. All known examples of nonflat noncompact homogeneous Einstein manifolds are isometric to solvable Lie groups endowed with left invariant Einstein metrics. It has been conjectured by D.V. Alekseevskii that every noncompact homogeneous Einstein manifold has maximal compact isotropy subgroups. This conjecture implies that the classification of noncompact homogeneous Einstein manifolds is reduced to the investigation of solvable Lie groups with left invariant Einstein metrics. The conjecture is still an open problem.

The purpose of this paper is to construct a class of noncompact homogeneous Einstein manifolds, which we call Boggino-Damek-Ricci type Einstein spaces (abbreviated to BDR-type Einstein spaces). Each element of this class is represented as a simply connected solvable Lie group with a left invariant metric. In 1985 J. Boggino constructed a class of Einstein manifolds with nonpositive sectional curvature which includes rank one symmetric spaces of noncompact type ([3]). These spaces are now called Damek-Ricci Einstein spaces ([2]). The class of BDR-type Einstein spaces is constructed as a 1-dimensional solvable extension of a 2-step nilpotent Lie algebra and contains Damek-Ricci Einstein spaces. Note that Damek-Ricci Einstein space has negative sectional curvature if and only if it is symmetric space ([3], [9]). In this paper we prove that there exist nonsymmetric BDR-type Einstein spaces with negative sectional curvature.

In Section 2 we define BDR-type spaces and investigate curvature property and Einstein condition of the BDR-type spaces. Using the Kaplan's J ([7]), we give formulas for curvature and Ricci transformation of BDR-type spaces (Lemma 2.1, 2.2). From Lemma 2.2 we see that the Einstein condition is reduced to the condition of the nilpotent part of BDR-type spaces (Proposition 2.3). We also give a sufficient condition that BDR-type space has nonpositive sectional curvature in Proposition 2.5. A Damek-Ricci Einstein space satisfies the condition of Proposition 2.5 and thus this gives another proof of the fact that it has nonpositive sectional curvature.

In Section 3 we construct BDR-type Einstein spaces which are not Damek-Ricci