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SIMPLIFIED PROBABILISTIC APPROACH TO THE HÖRMANDER THEOREM

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1. Introduction

In this paper, we shall discuss the problem to find sufficient conditions under which the probability law of the solution to the stochastic differential equation has a smooth transition density. There are many approaches to this problem in the theory of partial differential equations. It is well known that Hörmander ([2]) showed the relation between the hypoellipticity of second order elliptic differential operators and the dimension of the Lie algebra generated by vector fields associated with coefficients of the differential operator. Malliavin ([6], [7]) introduced the new differential calculus on the Wiener space, and applied its calculus to the probabilistic proof of the Hörmander theorem. He introduced the Ornstein-Uhlenbeck operator which is an unbounded self-adjoint non-negative operator on the L^2-space over the Wiener space, and obtained the integration by parts formula on this space ([3], [9]). On the other hand, Bismut ([1]) gave the different approach from Malliavin's work. He showed the integration by parts formula by using the Girsanov transformation. For the integration by parts formula, the integrability of the inverse of so called Malliavin covariance matrix is essential. Kusuoka and Stroock ([5]) presented a key lemma for the proof of the integrability. Norris ([8]) gave a simplified proof of the key lemma. His proof of it is still considerably long and complicated.

Instead of the Kusuoka-Stroock-Norris key lemma, we shall present a new lemma that plays an important role in the Malliavin calculus for SDE's. This can be proved easily and directly only by using simple stochastic calculations, that is, the Ito formula and the Fubini type theorem for stochastic integrals. In order to show the integrability of the inverse of the Malliavin covariance matrix, it suffices to prove the exponential decay of the Laplace transform of the quadratic form of the covariance matrix. It is possible to prove the exponential decay by an iterative application of the new lemma. Therefore, by using the new key lemma, we can easily show the Hörmander theorem.

The organization of this paper is as follows: in Section 2, we give some preliminaries that need in our argument, and introduce well-known results on the integrability of the inverse of the Malliavin covariance matrix. In Section 3, our main results are stated. In the final section, the Hörmander theorem is proved.