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A CHARACTERIZATION OF FOUR-GENUS OF KNOTS

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Introduction

We shall work in piecewise linear category. All knots and links will be assumed to be oriented in a 3-sphere S^3.

The 4-*genus* $g^*(K)$ of a knot K in $S^3 = \partial B^4$ is the minimum genus of orientable surfaces in B^4 bounded by K [1]. The *nonorientable* 4-*genus* $\gamma^*(K)$ is the minimum first Betti number of nonorientable surfaces in B^4 bounded by K [3]. For a slice knot, it is defined to be zero instead of one. The first author [4] defined the 4-*dimensional clasp number* $c^*(K)$ to be the minimum number of the double points of transversely immersed 2-disks in B^4 bounded by K. He gave an inequality $g^*(K) \leq c^*(K)$ [4, Lemma 9] and asked whether an equality $g^*(K) = c^*(K)$ holds or not. For this question, H. Murakami and the second author [3] gave an negative answer, i.e., they proved that there is a knot K such that $g^*(K) < c^*(K)$. Thus $c^*(K)$ is not enough to characterize $g^*(K)$. In this paper we give characterizations of 4-genus and nonorientable 4-genus by using certain 4-dimensional numerical invariants.

The local move as illustrated in Fig. 1(a) (resp. 1(b)) is called an H-*move* (resp. H'-*move*) for some positive integer n. Both an H-move and an H'-move realize a crossing change when $n = 1$. Thus these moves are certain kinds of unknotting operations of knots. Let L_n (resp. L'_n) be a link as illustrated in Fig. 2(a) (resp. 2(b)). Then we easily see that an H-move (resp. H'-move) can be realized by a *fusion/fission* [2, p. 95] of L_n (resp. L'_n); see Fig. 3. Therefore, for a knot K in ∂B^4, there is a *singular* disk D in B^4 with $\partial D = K$ that satisfies the following:

(1) D is a locally flat except for points $p_1, p_2, \ldots, p_{m(D)}$ in the interior of D.

(2) For each p_i ($i = 1, 2, \ldots, m(D)$) there is a small neighborhood $N(p_i)$ of p_i in B^4 such that $(\partial N(p_i), \partial(N(p_i) \cap D))$ is a link L_{n_i} (resp. L'_{n_i}) for some integer n_i.

We call these points $p_1, p_2, \ldots, p_{m(D)}$ *singularities of type* H (resp. *type* H'). Among these disks satisfying the above, $c^*_H(K)$ (resp. $c^*_{H'}(K)$) is the minimum number of $m(D)$. Note that $c^*_H(K) \leq c^*(K)$ and $c^*_{H'}(K) \leq c^*(K)$.

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