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AN AFFINE PROPERTY OF THE RECIPROCAL ASIAN OPTION PROCESS

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This note describes a property of the Asian option process, which neatly links the process at μ with the one at $-\mu$, where μ is the drift of the geometric Brownian motion. The proof is based on (i) a known result due to Yor, on the law of the Asian option process taken at an exponential time, and (ii) a recent result on beta and gamma distributions.

Suppose W is one-dimensional standard Brownian motion starting at the origin, and define what this author calls the *Asian option process*, for want of a better name:

$$A_t^{(\mu)} = \int_0^t e^{2\mu s + 2W_s} ds, \qquad t \geq 0, \quad \mu \in \mathbb{R}.$$

Asian options have payoffs such as $(A_t^{(\mu)} - K)_+$, and have been studied by numerous authors in Finance and Mathematics; reciprocal Asian options have payoffs such as $(K - 1/A_t^{(\mu)})_+$, and have not received much attention so far; for more details and references, the reader is referred to [3] and [4].

Theorem 1 ([6]). *Let T_λ be an exponentially distributed random variable, independent of W, with mean $1/\lambda$. Then*

$$2A_{T_\lambda}^{(\mu)} \stackrel{\mathcal{L}}{=} \frac{B_{1,\alpha}}{G_\beta},$$

where $B_{1,\alpha} \sim \mathrm{Beta}(1, \alpha)$ and $G_\beta \sim \Gamma(\beta, 1)$ are independent, $\alpha = \mu/2 + \sqrt{2\lambda + \mu^2}/2$, $\beta = \alpha - \mu$.

Theorem 2 ([2]). *For any $a, b, c > 0$,*

$$\frac{G_a}{B_{b,a+c}} + G'_c \stackrel{\mathcal{L}}{=} \frac{G_{a+c}}{B_{b,a}}.$$

where $G_a \sim \mathrm{Gamma}(a, 1)$, $G'_c \sim \mathrm{Gamma}(c, 1)$, $B_{b,a+c} \sim \mathrm{Beta}(b, a+c)$, $B_{b,a} \sim \mathrm{Beta}(b, a)$ and all variables are independent.