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AFFINE RULINGS OF NORMAL RATIONAL SURFACES

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Given an algebraic surface X satisfying:

(†) X is a complete normal rational surface, X is affine ruled and rank(Pic X_s) = 1,

where X_s denotes the smooth locus of X, consider:

Problem 1. Find all affine rulings of X.

Problem 2. Find all pairs of curves C_1, C_2 on X such that $X \setminus (C_1 \cup C_2)$ is isomorphic to $\mathbb{P}^2$ minus two lines.

Problem 3. Find all curves C in X such that $\bar{\kappa}(X_s \setminus C) = -\infty$.

This paper investigates Problem 1 for an arbitrary X satisfying (†). We define (Definition 1.14) the notion of a "basic" affine ruling of X and our main results describe how to construct all affine rulings of X, assuming that the basic ones are known. In the case where X is a weighted projective plane, the basic affine rulings of X are given in [6]; the present paper and [6] therefore constitute a solution to Problem 1 in that case.

Problem 3 (with $X = \mathbb{P}^2$) has been considered by several authors ([8], [9], [18], [19], [14]). In his review of [14] (see MR 82k:14013), M. H. Gizatullin mentions some unpublished examples found by V. I. Danilov and himself, and which seem to correspond to the list of basic affine rulings of $\mathbb{P}^2$. The case $X = \mathbb{P}^2$ was finally solved in [10]. Our generalization to weighted projective planes seems to be new, as well as our method—valid for any X satisfying (†)—which reduces the general problem to the determination of the *basic* affine rulings.

Let us briefly indicate how problems 1–3 are related to each other. Consider the stronger condition (‡) on a surface X:

(‡) X satisfies (†) and every singular point of X is a cyclic quotient singularity.

As an example, note that the weighted projective planes satisfy (‡) (they even satisfy Pic$(X_s) = \mathbb{Z}$; see [6] for these claims). Also note the following by-product of section 1: *A surface satisfying* (‡) *cannot have more than* 3 *singular points* (see Corollary 1.16).

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