A NOTE ON A CONJECTURE OF BRAUER

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To RICHARD BRAUER on the occasion of his 60th Birthday

§ 1. Introduction

In [1] R. Brauer asked the following question: Let $\mathfrak{G}$ be a finite group, p a rational prime number, and B a p-block of $\mathfrak{G}$ with defect d and defect group $\mathfrak{D}$. Is it true that $\mathfrak{D}$ is abelian if and only if every irreducible character in B has height 0? The present results on this problem are quite incomplete. If $d=0, 1, 2$ the conjecture was proved by Brauer and Feit, [4] Theorem 2. They also showed that if $\mathfrak{D}$ is cyclic, then no characters of positive height appear in B. If $\mathfrak{D}$ is normal in $\mathfrak{G}$, the conjecture was proved by W. Reynolds and M. Suzuki, [12]. In this paper we shall show that for a solvable group $\mathfrak{G}$, the conjecture is true for the largest prime divisor p of the order of $\mathfrak{G}$. Actually, one half of this has already been proved in [7]. There it was shown that if $\mathfrak{G}$ is a p-solvable group, where p is any prime, and if $\mathfrak{D}$ is abelian, then the condition on the irreducible characters in B is satisfied.

The proof of the converse presented here is somewhat difficult. A series of reductions gives rise to the following situation: $\mathfrak{G}$ is a finite solvable group of order pg', where $(p, g')=1$, such that $\mathfrak{G}$ has no proper normal subgroups of p'-index. Moreover $\mathfrak{G}$ acts faithfully and irreducibly on a vector space $\mathcal{V}$ over a finite field, such that each vector v in $\mathcal{V}$ is fixed by some Sylow p-subgroup of $\mathfrak{G}$. Using methods similar to those used by Huppert in [10], [11], we shall see that $g'=1$ if p is the largest prime divisor of the order of $\mathfrak{G}$.

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