A CHARACTERIZATION OF INVARIANT AFFINE CONNECTIONS

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1. Introduction and statement of theorem. 1. In [1] Ambrose and Singer gave a necessary and sufficient condition (Theorem 3 here) for a simply connected complete Riemannian manifold to admit a transitive group of motions. Here we shall give a simple proof of a more general theorem—Theorem 1 (the proof of Theorem 1 became suggestive to us after we noted that the T_x of [1] is just the a_x of [6] when X is restricted to $\mathfrak{p}_0$, see [6], p. 539). In fact after introducing, below, the notion of one affine connection A on a manifold being rigid with respect to another affine connection B on M and making some observations concerning such a relationship, Theorem 1 is seen to be a reformulation of Theorem 2. But Theorem 2 may be obtained as a consequence of some work of Nomizu and Kobayashi.[1] Here we refer especially to the work of these mathematicians on the theory of affine connections which are invariant under parallelism. Such an affine connection was called locally reductive in [4]. (Since a slight elaboration of this theory was needed for our purposes, rather than "fill in", we have preferred instead to give a somewhat different, almost self-contained, account of the relevant portion of this theory here.)

Without any statement to the contrary it will be assumed throughout this paper that the manifold M and any affine connection or tensor field to be considered on M is of class C^∞.

1.2. We shall need some definitions.

(a) The notion of a reductive homogeneous space was introduced by Nomizu [7]. Assume that G is a connected Lie group and G is given as operating transitively on a manifold M as a group of homeomorphisms in such a way that the map $G \times M \to M$ defined by $(g, o) \to g \cdot o$ is of class C^∞. Here $g \cdot o$ is the image of $o \in M$ under the action of $g \in M$. Let $\mathfrak{g}$ be the Lie algebra of

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[1] We have been informed by correspondence that Nomizu has also obtained a similar generalization of Theorem 3.