NOTE ON RELATIVE HOMOLOGICAL DIMENSION

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Let R be a ring with identity element 1, and let S be a subring of R containing 1. We consider R-modules on which 1 acts as the identity map, and we shall simultaneously regard such R-modules as S-modules in the natural way. In [4], we have defined the relative analogues $\mathrm{Ext}^n_{(R,S)}$ of the functors Ext^n_R of Cartan-Eilenberg [1], and we have briefly treated the corresponding relative analogues of module dimension and global ring dimension. If M is an R-module the relative projective dimension of M is denoted $d_{R,S}(M)$. It is the smallest non-negative integer n (or ∞) for which there is an R-module N such that $\mathrm{Ext}^n_{(R,S)}(M, N) \neq (0)$. The relative global dimension $d(R, S)$ of the pair (R, S) is defined as $\sup_M (d_{R,S}(M))$. We use the similar notations $d_R(M)$ and $d(R)$ for the absolute projective R-module dimension of M and the global dimension of R, respectively.

Our purpose here is to establish some elementary relations between the relative dimensions and the absolute dimensions, and to point out how the relative dimensions can be used to obtain information on absolute dimensions. In particular, we are thereby led to a simple derivation of the known results on the dimension of polynomial rings. In this connection, I have had a number of clarifying discussions with Maurice Auslander, and I wish to thank him here for his valuable comments.

§1. Following Cartan-Eilenberg, we shall say that R is right S-flat if $\mathrm{Tor}^n_S(R, C) = (0)$, for every S-module C and all $n > 0$, or, equivalently, if, for every monomorphism $U \to V$ of S-modules, the induced homomorphism $R \otimes_S U \to R \otimes_S V$ is a monomorphism. In particular, R is right S-flat whenever it is S-projective as a right S-module.

THEOREM 1. *Suppose that R is right S-flat. Then, for every R-module M,*

$$d_R(M) \leqq d_{R,S}(M) + d(S).$$

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