ON THE DIMENSION OF MODULES AND ALGEBRAS, X

A RIGHT HEREDITARY RING WHICH IS NOT LEFT HEREDITARY

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A ring R is said to be right (left) hereditary if every right (left) ideal in R is projective, that is, a direct summand of a free R-module. Cartan and Eilenberg [3, p. 15] ask whether there exists a right hereditary ring which is not left hereditary. The answer: yes.

THEOREM. *Let V be a vector space of countably infinite dimension over a field F. Let C be the algebra of all linear transformations on V with finite-dimensional range. Let B be the algebra obtained by adjoining a unit element to C. Let $A = B \otimes B$ (all tensor products are over F). Then A is right hereditary but not left hereditary.*

The proof that A is right hereditary is broken into four lemmas.

LEMMA 1. *Let R be a regular ring (for any a there exists an x such that $axa = a$). Then every countably generated right ideal I in R is projective.*

Proof. It is known that any finitely generated right ideal in R can be generated by an idempotent. Hence I can be expressed as the union of an ascending sequence of right ideals generated by idempotents. Each of these is projective and is a direct summand of its successor. Hence I is a direct sum of projective ideals and is itself projective.

LEMMA 2. *Let R be a ring, J a two-sided ideal in R. Suppose that in J and R/J every right ideal is countably generated. Then the same is true in R.*

Proof. Take a right ideal I in R. Using $*$ for image mod J, we pick a countable set $\{a_n^*\}$ of generators for I^*. Pick elements $a_n \in I$ mapping on a_n^*.

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