ON THE DIMENSION OF MODULES AND ALGEBRAS IX

DIRECT LIMITS

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Let J be a directed set and let $\{\Lambda_j, \varphi_{ij}\}$ be a direct system of rings indexed by J and with limit Λ. Let $\{A_j, \psi_{ij}\}$ be a direct system of groups indexed by J. Assume that each A_j is a left Λ_j-module and that $\psi_{ij}(\lambda a) = \varphi_{ij}(\lambda)\psi_{ij}(a)$ for each $\lambda \in \Lambda_j$, $a \in A_j$. Then the limit A of $\{A_j, \psi_{ij}\}$ is a left Λ-module.

THEOREM. *If J is countable, then*

$$\text{l.}\dim_\Lambda A \leqq 1 + \sup_J \text{l.}\dim_{\Lambda_j} A_j.$$

COROLLARY 1. *If J is countable, then*

$$\text{l.gl.}\dim \Lambda \leqq 1 + \sup_J \text{l.gl.}\dim \Lambda_j.$$

COROLLARY 2. *Let $\{K_j, \nu_{ij}\}$ be a direct system of commutative rings indexed by J and with limit K. Assume that each Λ_j is a K_j-algebra and that $\varphi_{ij}(k\lambda) = \nu_{ij}(k)\varphi_{ij}(\lambda)$ for $k \in K_j$, $\lambda \in \Lambda_j$. Then Λ is a K-algebra. If J is countable, then*

$$K\text{-}\dim \Lambda \leqq 1 + \sup. K_j\text{-}\dim \Lambda_j.$$

To derive Cor. 2 we note that

$$K\text{-}\dim \Lambda = \text{l.}\dim_{\Lambda^e} \Lambda, \quad \text{where} \quad \Lambda^e = \Lambda \otimes_K \Lambda^*,$$

and that Λ^e is the direct limit of $\{\Lambda_j^e\}$. Cor. 2 is a generalization of a theorem by Kuročkin [1] (see also [2], p. 92).

Proof of the Theorem. We consider the exact sequences

$$0 \longrightarrow R_j \longrightarrow F_j \longrightarrow A_j \longrightarrow 0, \qquad j \in J$$
$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$$

where F_j is the free Λ_j-module with the elements of A_j as Λ_j-basis and

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