A GLOBAL FORMULATION OF THE FUNDAMENTAL THEOREM OF THE THEORY OF SURFACES IN THREE DIMENSIONAL EUCLIDEAN SPACE

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§ 1. Introduction

Let us consider a surface S of class C^3 in Euclidean space E_3 and

$$x^i = x^i(u^1, u^2) \qquad (i, j, k = 1, 2, 3) \tag{1.1}$$

be its parametric representation of class C^3. Then

$$g_{\beta\gamma} = \sum_i \frac{\partial x^i}{\partial u^\beta} \frac{\partial x^i}{\partial u^\gamma} \qquad (\alpha, \beta, \gamma = 1, 2) \tag{1.2}$$

$$h_{\beta\gamma} = \left| \frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma} \quad \frac{\partial x^i}{\partial u^1} \quad \frac{\partial x^i}{\partial u^2} \right| \Big/ (g_{11} g_{22} - g_{12}^2) \tag{1.3}$$

are the first and second fundamental tensors of the surface S. If we put

$$\frac{\partial x^i}{\partial u^\alpha} = x^i_\alpha \tag{1.4}$$

and denote the unit normal vector of S by e^i, then the functions x^i satisfy the derived equations

$$\begin{aligned} \frac{\partial x^i_\beta}{\partial u^\gamma} &= \left\{ {\alpha \atop \beta\gamma} \right\} x^i_\alpha + h_{\beta\gamma} e^i, && \text{(Gauss)} \\ \frac{\partial e^i}{\partial u^\beta} &= - h^\alpha_\beta x^i_\alpha, && \text{(Weingarten)} \end{aligned} \tag{1.5}$$

where $\left\{ {\alpha \atop \beta\gamma} \right\}$'s are Christoffel's symbols constructed from $g_{\beta\gamma}$ and

$$h^\alpha_\beta = g^{\alpha\gamma} h_{\beta\gamma}. \tag{1.6}$$

By virtue of the relation

$$\frac{\partial^2 x^i_\beta}{\partial u^\gamma \partial u^\delta} = \frac{\partial^2 x^i_\beta}{\partial u^\delta \partial u^\gamma}, \qquad \frac{\partial^2 e^i}{\partial u^\beta \partial u^\gamma} = \frac{\partial^2 e^i}{\partial u^\gamma \partial u^\beta}, \tag{1.7}$$

we see that $g_{\beta\gamma}$ and $h_{\beta\gamma}$ satisfy the following relations:

Received November 18, 1957.
Revised January 27, 1958.