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A CHARACTERIZATION OF THE FINITE SIMPLE GROUPS PSp(4, q), G_2(q), D_4^2(q), I

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Suppose that G is the projective symplectic group $PSp(4, q)$, the Dickson group $G_2(q)$, or the Steinberg "triality-twisted" group $D_4^2(q)$, where q is an odd prime power. Then G is a finite simple group, and G contains an involution j such that the centralizer $C(j)$ in G has a subgroup of index 2 which contains j and which is the central product of two groups isomorphic with $SL(2, q_1)$ and $SL(2, q_2)$ for suitable q_1, q_2. We wish to show that conversely the only finite simple groups containing an involution with this property are the groups $PSp(4, q)$, $G_2(q)$, $D_4^2(q)$. In this first paper we shall prove the following result.

THEOREM. *Let G be a finite group with subgroups L_1, L_2 such that $L_1 \simeq SL(2, q_1)$, $L_2 \simeq SL(2, q_2)$, $[L_1, L_2] = 1$, $L_1 \cap L_2 = \langle j \rangle$, where j is an involution, and $|C(j): L_1L_2| = 2$. Suppose that $G \neq C(j)O(G)$. Then one of the following holds:*

(a) *$q_1 = q_2$, and L_1, L_2 are not normal in $C(j)$.*

(b) *$q_1 = q_2$, and L_1, L_2 are both normal in $C(j)$.*

(c) *One of the numbers q_1, q_2 is the cube of the other.*

Furthermore, in each case, $C(j)$ is uniquely determined to within isomorphism.

Here $O(G)$ denotes the largest normal subgroup of odd order in G, and the condition $G \neq C(j)O(G)$ is obviously satisfied if G is simple. The groups $PSp(4, q)$, $G_2(q)$, $D_4^2(q)$ satisfy the hypotheses of the theorem, and belong to the cases (a), (b), (c) *respectively. By the uniqueness statement of the theorem, $C(j)$ is isomorphic with the centralizer of an involution in $PSp(4, q)$, $G_2(q)$ or $D_4^2(q)$, where $q = \min\{q_1, q_2\}$. In case* (a) *it follows that G must be isomorphic with $PSp(4, q)$* [18]. *In the sequel to this paper it will be shown that, in cases* (b), (c), *G must be isomorphic with $G_2(q)$, $D_4^2(q)$ respectively* [12].

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