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SOME CONDITIONS UNDER WHICH SEQUENCES OF FUNCTIONS ARE UNIFORMLY BOUNDED

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§1. Introduction

After one introduces the theory of normal families in a course in complex analysis, the usual pattern is to give an example of a non-normal family. One of the simplest, of course, is the sequence $f_n(z) = nz$, $n = 1, 2, \cdots$. The very devastating effect of multiplying by zero insures the required abnormality! If one asks for a slightly more sophisticated example, we offer $f_n(z) = \frac{e^{nz}}{n}$, $n = 1, 2 \cdots$; here the $f_n(z)$ are zero free. However, the difference in behavior between the sequence $\left\{|f_n(0)| = \frac{1}{n}\right\}$ and $|f_n(z)| = \frac{e^{nx}}{n}$ $z = x + iy \neq 0$ is obvious.

In this paper we offer an explanation for this state of affairs. To be precise given a sequence $\{f_n\}$ of bounded holomorphic functions defined in the unit disk D in the complex plane we establish criteria based upon a comparison of $\{|f_n(0)|\}$ and $\{M(f_n)\}$, $M(f_n) = \max_{|z|<1} |f_n(z)|$ insuring that the sequence $\{f_n\}$ will be uniformly bounded on certain compact subsets of D. We then extend the result in several directions to entire and harmonic functions.

Let $D(r)$, $0 < r < \infty$, denote the open disk in the complex plane with centre at the origin and radius r, while $\bar{D}(r)$ indicates the closure of $D(r)$; and $D(\infty)$ indicates the finite complex plane. If f is a complex valued function defined in $D(R)$, for $0 \leq r \leq R$ let

$$M(r, f) = \sup_{|z|<r} |f(z)|$$

and

$$m(r, f) = \inf_{|z|<r} |f(z)|.$$

If E is a set contained in $D(R)$ set

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