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Nagoya Math. J.
Vol. 69 (1978), 131-138

ON A THEOREM OF RAMANAN

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Let G be a simply connected Lie group and P a parabolic subgroup without simple factor. A finite dimensional irreducible representation of P defines a homogeneous vector bundle E over the homogeneous space G/P. Ramanan [2] proved that, if the second Betti number b_2 of G/P is 1, the inequality in Definition (2.3) holds provided F is locally free. Since the notion of the H-stability was not established at that time, it was inevitable to assume that $b_2 = 1$ and F is locally free. In this paper, pushing Ramanan's idea through, we prove that E is H-stable for any ample line bundle H. Our proof as well as Ramanan's depends on the Borel-Weil theorem. If we recall that the Borel-Weil theorem fails in characteristic $p > 0$, it is interesting to ask whether our theorem remains true in characteristic $p > 0$.

§ 1. The Borel-Weil theorem

Let us review the Borel-Weil theorem on which the proof of our theorem heavily depends. We use the notation of Kostant [1] with slight modifications. For example, we shall denote by $\mathfrak{p}$ a parabolic Lie subalgebra which Kostant denotes by $\mathfrak{u}$. In this section all the results are stated without proofs. The details are found in the paper of Kostant cited above.

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and let $(\mathfrak{g})$ be the Cartan-Killing form on $\mathfrak{g}$ namely $(x, y) = \mathrm{tr}\,(adx \circ ady)$ for $x, y \in \mathfrak{g}$.

A compact form of $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ satisfying the following conditions:

(i) $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ is the direct sum of real Lie algebra.

(ii) the Cartan-Killing form is negative definite on $\mathfrak{k}$. We fix a compact form once and for all. Let $\mathfrak{q} = i\mathfrak{k}$ so that the restriction of the Cartan-Killing form to $\mathfrak{q}$ is positive definite. Evidently we have a real decom-

Received February 21, 1977.