T. Kubota
Nagoya Math. J.
Vol. 61 (1976), 113-116

A COMPLEX AIRY INTEGRAL

Dedicated to Professor Tikao Tatuzawa on his 60th birthday

TOMIO KUBOTA

The Airy integral is a formula concerning the Fourier transform of a function like $\sin x^3$ or $\cos x^3$, and is written, for instance in [2], as

$$\int_0^\infty \cos(t^3 - xt)dt = \frac{1}{3}\pi\sqrt{\frac{1}{3}x}\left[J_{-1/3}\left(\frac{2x\sqrt{x}}{3\sqrt{3}}\right) + J_{1/3}\left(\frac{2x\sqrt{x}}{3\sqrt{3}}\right)\right]$$

for $x \geqq 0$.

In this paper, we shall prove a similar formula

$$(1) \quad \int_C e(z^3 - 3zw)dV(z) = \frac{1}{3}\pi^2\left(\sin\frac{\pi}{3}\right)^{-1}|w|(|J_{-1/3}(2\pi w^{3/2})|^2 - |J_{1/3}(2\pi w^{3/2})|^2)$$

containing same Bessel functions and the exponential function $e(z) = \exp(\pi\sqrt{-1}(z + \bar{z}))$, where $dV(z)$ is the usual euclidean measure, and the integral $\int_C$ should be interpreted as $\lim_{Y\to\infty}\int_{|z|<Y}$. This is a byproduct of the results in [1].

The proof of our main result (1) is reduced to an equality between Mellin transforms of certain functions. Let us first treat the purely computational part of the proof. If $z = r\exp(\sqrt{-1}\theta)$ and $w = r'\exp(\sqrt{-1}\theta')$, $(r, r' \geqq 0,\ \theta, \theta' \in \boldsymbol{R})$, are polar expressions of complex numbers z and w, then a general formula on the Bessel function J_m says

$$e(z) = \sum_{m=-\infty}^{\infty} \sqrt{-1}^m J_m(2\pi r)\exp(\sqrt{-1}m\theta) .$$

This implies

$$e(z^3) = \sum_{m=-\infty}^{\infty} \sqrt{-1}^m J_m(2\pi r^3)\exp(\sqrt{-1}3m\theta) ,$$

$$e(-3zw) = \sum_{m=-\infty}^{\infty} (-\sqrt{-1})^m J_m(6\pi rr')\exp(\sqrt{-1}m(\theta + \theta')) ,$$

Received June 2, 1975.
This research is supported by NSF Grant GP-43950 (SK-CUCB).