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ON A FUNCTION ANALOGOUS TO $\log\eta(\tau)$

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1. Introduction

Let us denote by $\eta(z)$ the classical η-function of Dedekind defined by

$$\eta(z) = e^{\pi i z/12}\prod_{n=1}^{\infty}(1 - e^{2\pi i n z}) , \qquad \operatorname{Im}(z) > 0 .$$

If $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$, then the classical law of transformation of $\log\eta(z)$ asserts that if $\sigma(z) = (az + b)/(cz + d)$, then

$$\begin{aligned}\log\eta(\sigma(z)) &= \log\eta(z) + \frac{\pi i b}{12} \quad (c = 0) \\ &= \log\eta(z) + \frac{1}{2}\log\left(\frac{cz + d}{i}\right) + \pi i\frac{a + d}{12c} - \pi i s(d, c) \quad (c > 0)\end{aligned} \tag{1}$$

where all logarithms are taken with respect to the principal branch and

$$s(d, c) = \sum_{\mu \ (\mathrm{mod}\ c)} \left(\left(\frac{d\mu}{c}\right)\right)\left(\left(\frac{\mu}{c}\right)\right)$$

and where

$$((x)) = x - [x] - \tfrac{1}{2} \qquad \text{if } x \text{ is not an integer,}$$
$$0 \qquad \text{otherwise.}$$

The sum $s(d, c)$ is called a Dedekind sum, and appears in many number-theoretic investigations.

In [1], we have introduced a generalization of the function $\eta(z)$ associated to a totally real algebraic number field K. Our generalization arose from a generalization of Kronecker's second limit formula. Moreover, we showed in [1] that a classical conjecture of Hecke concerning class numbers of algebraic number field could be reduced to determining

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