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MODULAR FORMS OF HALF INTEGRAL WEIGHT AND THE INTEGRAL OF CERTAIN THETA-FUNCTIONS

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§ 0. Introduction.

Recently G. Shimura [1] constructed modular forms of integral weight from the forms of half integral weight. His construction is rather indirect. Indeed, he proved that the Dirichlet series, obtained from a form of half integral weight, multiplied by a certain L-function, corresponds to a modular form of an integral weight by means of the characterization of modular forms due to Weil.

In this paper, we shall give a more direct method of constructing modular forms of integral weight, using Siegel-Weil's indefinite theta series, and at the same time prove the conjecture related to the level of such forms (the former part of (A) in § 4 of [1]). The relation of the theory of the theta series and Shimura's result was first pointed out by T. Shintani [2]. Indeed, he constructed, in contrast to Shimura's result modular forms of half integral weight, using an indefinite theta series, and showed that this correspondence is almost reciprocal to Shimura's. We note that our use of the theta series is, however, different from Shintani's.

Let N be a positive integer, χ a character modulo $4N$ and $\chi_1 = \chi\left(\frac{-1}{*}\right)^{\lambda}$ with a positive integer λ. We denote by H the complex upper half plane, and by $x = (x_1, x_2, x_3)$ an element of the vector space $\boldsymbol{R}^3$. For $g \in SL(2, \boldsymbol{R})$ we define a function on $\boldsymbol{R}^3$ by $f(x) = (x_1 - ix_2 - x_3)^{\lambda} \exp((-2\pi/N)(2x_1^2 + x_2^2 + 2x_3^2))$. For $\kappa = 2\lambda + 1$, $z = u + iv \in H$ and for the lattice $L' = \boldsymbol{Z} \oplus N\boldsymbol{Z} \oplus (N\boldsymbol{Z}/4)$ in $\boldsymbol{Q}^3$, we define a theta series $\theta(z, g)$ by

$$\theta(z, g) = \sum_{x \in L'} \bar{\chi}_1(x_1) v^{(3-\kappa)/4} (\exp(2\pi i(u/N)(x_2^2 - 4x_1x_3))) f(\sqrt{v}\, g^{-1}x) ,$$

where $\sqrt{v} \in \boldsymbol{R}$ is viewed as a scalar of the vector space $\boldsymbol{R}^3$, and

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