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ON A CLASSIFICATION OF THE FUNCTION FIELDS OF ALGEBRAIC TORI

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Let Π be a finite group and denote by $\boldsymbol{M}_{\Pi}$ the class of all (finitely generated Z-free) Π-modules. In the previous paper [3] we defined an equivalence relation in $\boldsymbol{M}_{\Pi}$ and constructed the abelian semigroup $T(\Pi)$ by giving an addition to the set of all equivalence classes in $\boldsymbol{M}_{\Pi}$. The investigation of the semigroup $T(\Pi)$ seems interesting and important, because this gives a classification of the function fields of algebraic tori defined over a field k which split over a Galois extension of k with group Π.

The purpose of this paper is to obtain information on the structure of the semigroup $T(\Pi)$.

We will recall the definitions given in [2] and [3]. A Π-module is called a permutation Π-module if it can be expressed as a direct sum of $\{Z\Pi/\Pi_i\}$ where each Π_i is a subgroup of Π. Further a Π-module M is called a quasi-permutation Π-module if there exists an exact sequence $0 \to M \to S \to S' \to 0$ where S and S' are permutation Π-modules. The dual module $\mathrm{Hom}_Z(M, Z)$ of a Π-module M is denoted by M^*. The augmentation ideal of $Z\Pi$ is denoted by I_{Π} and the dual module I_{Π}^* of I_{Π} is called the Chevalley's module of Π ([1], [2]).

Let k be a field. Let K be a Galois extension of k with group $\cong \Pi$ and let M be a Π-module with a Z-free basis $\{u_1, u_2, \cdots, u_n\}$. Define the action on the rational function field $K(X_1, X_2, \cdots, X_n)$ with n variables $X_1, X_2, \cdots, X_n$ over K by putting, for each $\sigma \in \Pi$ and $1 \leqq i \leqq n$, $\sigma(X_i) = \prod_{j=1}^{n} X_j^{m_{ij}}$ when $\sigma \cdot u_i = \sum_{j=1}^{n} m_{ij} u_j$, $m_{ij} \in Z$, and denote by $K(M)$ $K(X_1, X_2, \cdots, X_n)$ with this action of Π. It is well known ([7]) that there is a duality between the category of all algebraic tori defined over k which split over K and the category of all Π-modules. In fact, if T is an algebraic torus defined over k which splits over K, then the character

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