T. Ono
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ON SOME RATIONALITY PROPERTIES OF HOPF MAPS

TAKASHI ONO

As is well known, the Hopf fibration $S^3 \to S^2$ is the restriction of the map $h: \boldsymbol{R}^4 \to \boldsymbol{R}^3$ given by $h(x) = (x_1^2 + x_2^2 - x_3^2 - x_4^2, 2(x_1x_3 - x_2x_4), 2(x_1x_4 + x_2x_3))$, a system of three quadratic forms [2]. Since spheres and the map are defined by polynomials with coefficients in $\boldsymbol{Q}$, the original setting can be considered as a localization at infinity of the underlying algebraic sets and morphism defined over $\boldsymbol{Q}$. This makes one think of the arithmetic of the Hopf maps.

The purpose of this note is to arrange some basic materials so that one can see the algebraic mechanism of the Hopf's construction of quadratic maps mapping sphere to sphere and to prove a single Theorem concerning the surjectivity of the Hopf maps defined over an algebraic number field.

We shall use freely the results on a single quadratic form in [4]. As for elementary properties of the Clifford algebra, see [1].

§ 1. Involutorial representation of Clifford algebras

Let K be a field of characteristic not two and (V, q) be a non-degenerate quadratic space over K. We shall denote by $C(q)$ the Clifford algebra of (V, q). We consider V, as usual, as a subspace of $C(q)$. The algebra $C(q)$ has a unique involution $c \mapsto \bar{c}$ such that $\bar{v} = -v$ for all $v \in V$.

Let (Y, φ) be another non-degenerate quadratic space over K. We denote by Φ the symmetric bilinear form on Y associated to φ, i.e. $\Phi(y_1, y_2) = \frac{1}{2}(\varphi(y_1 + y_2) - \varphi(y_1) - \varphi(y_2))$, $y_1, y_2 \in Y$. The quadratic form φ defines an involution $a \mapsto a^*$ of the algebra $E = \operatorname{End} Y$ of K-endomorphisms of the space Y by the rule

$$\Phi(ay_1, y_2) = \Phi(y_1, a^*y_2)\ , \qquad a \in E\ . \tag{1.1}$$

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