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THE VARIATIONAL THEORY OF HIGHER-ORDER LINEAR DIFFERENTIAL EQUATIONS

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§1. Introduction

In his paper [2], [3], D. A. Hejhal investigated the variational theory of linear polynomic functions. In this paper we are concerned with the variational theory of higher-order differential equations. To be more precise, consider a compact Riemann surface having genus $g > 1$. As is well known, we can choose a projective coordinate covering $\mathfrak{A} = (U_\alpha, z_\alpha)$. Fix this coordinate covering of X. We shall be concerned with linear ordinary differential operators of order n defined in each projective coordinate open set U_α

$$L_{n,\alpha}(P_\alpha | z_\alpha) = \left(\frac{d}{dz_\alpha}\right)^n + \sum_{\ell=1}^{n} \binom{n}{\ell} P_{n,\alpha}(z_\alpha)\left(\frac{d}{dz_\alpha}\right)^{n-\ell} \tag{1.1}$$

where coefficients $P_{1,\alpha}(z_\alpha), \cdots, p_{n,\alpha}(z_\alpha)$ are holomorphic in U_α. Differential operators $\{L_{n,\alpha}(P_\alpha | z_\alpha)\}$ are called a semi-canonical form if $P_{1,\alpha}(z_\alpha) = 0$ for all α.

Let $\lambda \in H^1(X, \mathcal{O}_x)$ be a complex line bundle on X. Differential operators $\{L_{n,\alpha}(P_\alpha | z_\alpha)\}$ are called λ-related if in each intersection $U_\alpha \cap U_\beta$

$$L_{n,\alpha}(P_\alpha | z_\alpha)y = \left(\frac{dz_\beta}{dz_\alpha}\right)^n \lambda_{\alpha\beta}(z)^{-1} L_{n,\beta}(P_\beta | z_\beta)\lambda_{\alpha\beta}(z_\beta)y\,. \tag{1.2}$$

We shall prove an analogous theorem of the Laguerre-Forsyth's basic differential invariants.

THEOREM 1.1. *Let $\{L_{n,\alpha}(P_\alpha | z_\alpha)\}$ be a λ-related semi-canonical form, then*

$$(\theta_{m,\alpha}(z_\alpha)) \in \Gamma(\mathfrak{A}, \mathcal{O}(\kappa^m)) \qquad (m = 2, 3, \cdots, n)$$

where $\theta_{m,\alpha}(z_\alpha)$ is holomorphic function in U_α defined by:

$$\theta_{m,\alpha}(z_\alpha) = \frac{1}{2}\sum_{k=0}^{m-2}(-1)^k \frac{(m-2)!\,m!\,(2m-k-2)!}{(m-k-1)!\,(m-k)!\,(2m-3)!\,k!}\left(\frac{d}{dz_\alpha}\right)^k P_{m-k,\alpha}(z_\alpha)\,.$$

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