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ON A CERTAIN HOMOLOGY OF FINITE PROJECTIVE SPACES

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1. We denote by $\boldsymbol{P}^n(q)$ the projective space of dimension n over a finite field $GF(q)$ with q elements, and we mean by an i-flat a linear subspace of dimension i in $\boldsymbol{P}^n(q)$. Denoting

$$L_i = L_i(\boldsymbol{P}^n(q)) = \{i\text{-flats in } \boldsymbol{P}^n(q)\}\,,$$
$$\boldsymbol{Z}_{q+1} = \boldsymbol{Z}/(q+1)\boldsymbol{Z}\,,$$
$$C_{i+1}(\boldsymbol{P}^n(q), \boldsymbol{Z}_{q+1}) = \{\sum_{\sigma\in L_i} a_\sigma\sigma \,|\, a_\sigma \in \boldsymbol{Z}_{q+1}\}\,,$$
$$C_{i+1}(\boldsymbol{P}^n(q), GF(l)) = \{\sum_{\sigma\in L_i} a_\sigma\sigma \,|\, a_\sigma \in GF(l)\}$$
$$C_0(\boldsymbol{P}^n(q), \boldsymbol{Z}_{q+1}) = \boldsymbol{Z}_{q+1}\,,\quad C_0(\boldsymbol{P}^n(q), GF(l)) = GF(l) \quad (l|q+1)\,,$$

we have chain complexes

$$\begin{aligned} &C_{n+1}(\boldsymbol{P}^n(q), \boldsymbol{Z}_{q+1}) \xrightarrow{\partial_{n+1}} C_n(\boldsymbol{P}^n(q), \boldsymbol{Z}_{q+1}) \xrightarrow{\partial_n} \cdots \\ &\qquad \xrightarrow{\partial_2} C_1(\boldsymbol{P}^n(q), \boldsymbol{Z}_{q+1}) \xrightarrow{\partial_1} \boldsymbol{Z}_{q+1} \\ &= C_0(\boldsymbol{P}^n(q), \boldsymbol{Z}_{q+1}) \xrightarrow{\partial_0} \{0\} \end{aligned} \tag{1}$$

$$\begin{aligned} &C_{n+1}(\boldsymbol{P}^n(q), GF(l)) \xrightarrow{\partial_{n+1}} C_n(\boldsymbol{P}^n(q), GF(l)) \xrightarrow{\partial_n} \cdots \\ &\qquad \xrightarrow{\partial_2} C_0(\boldsymbol{P}^n(q), GF(l)) \xrightarrow{\partial_1} GF(l) \\ &= C_0(\boldsymbol{P}^n(q), GF(l)) \xrightarrow{\partial_0} \{0\}\,, \end{aligned} \tag{2}$$

where the boundary operators ∂_i are defined as follows,

$$\begin{cases} \partial_{i+1}(\sum_{\sigma\in L_i} a_\sigma\sigma) = \sum_{\sigma\in L_i} a_\sigma(\sum_{\substack{\tau\in L_{i-1}\\ \tau\subset\sigma}} \tau) & (i\geq 1) \\ \partial_1(\sum_{\sigma\in L_0} a_\sigma\sigma) = \sum_{\sigma\in L_0} a_\sigma\,. \end{cases}$$

In fact $\partial_i \circ \partial_{i+1} = 0$, because for each pair $\sigma \supset \lambda (\sigma \in L_i, \lambda \in L_{i-2})$

$$|\{\tau \in L_{i-1} | \lambda \subset \tau \subset \sigma\}| = q + 1$$

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