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THE BOUNDARY BEHAVIOUR OF HADAMARD LACUNARY SERIES

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§ 1. Introduction

A convergent power series $f(z)$ in the open unit disk $\boldsymbol{D}$ is called Hadamard lacunary if it is expressed as follows:

$$(1) \qquad f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}, \quad n_{k+1}/n_k \geq q \ (k \geq 1) \qquad \text{for some } q > 1 .$$

We shall discuss the boundary behaviour of Hadamard lacunary series. For a subset X of $\boldsymbol{D}$, we put $b(X) = \overline{X} \cap \partial\boldsymbol{D}$, where $\overline{X}$ is the closure of X and $\partial\boldsymbol{D}$ the boundary of $\boldsymbol{D}$. We say that an analytic function $g(z)$ in $\boldsymbol{D}$ has an extended complex number ω as an asymptotic value if there exists a path $\gamma \subset \boldsymbol{D}$ with $b(\gamma) \neq \emptyset$ such that $\lim_{|z|\to 1, z\in\gamma} g(z) = \omega$. We say that $g(z)$ has an asymptotic value at $a \in \partial\boldsymbol{D}$ if there exists a path $\gamma \subset \boldsymbol{D}$ with $b(\gamma) = \{a\}$ such that $\lim_{z\to a, z\in\gamma} g(z)$ exists. The Maclane class $\mathscr{A}$ is the totality of analytic functions $g(z)$ in $\boldsymbol{D}$ such that $g(z)$ has asymptotic values at a dense subset of $\partial\boldsymbol{D}$.

In [5], G. R. Maclane proved that a power series $f(z)$ given by (1) with $q > 3$ belongs to $\mathscr{A}$. It is conjectured that Hadamard lacunary series belong to $\mathscr{A}$. In [1], J. M. Anderson noted that Maclane's result is deduced from a result of K. G. Binmore in [2]. In [3], K. G. Binmore and R. Hornblower gave an another partial answer to this question. We shall answer this question. The main purpose of this paper is to show

THEOREM. *Let $f(z)$ be an Hadamard lacunary series given by* (1) *with* $\limsup_{k\to\infty} |c_k| = \infty$. *Then $f(z)$ has an asymptotic value ∞ at every point of $\partial\boldsymbol{D}$.*

It is known that the Hadamard lacunary series in our theorem has no finite asymptotic value ([2]), and hence ∞ is a unique asymptotic value.

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