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REMARKS ON EXTREMAL KÄHLER METRICS ON RULED MANIFOLDS

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Introduction

Let X be a compact Kähler manifold and γ a Kähler class. For a Kähler metric g on X we denote by R_g the scalar curvature on X According to Calabi [3][4], consider the functional $\Phi(g) = \int R_g^2 dv_g$ defined on the set of all the Kähler metrics g whose Kähler forms belong to γ, where dv_g is the volume form associated to g . Such a Kähler metric is called *extremal* if it gives a critical point of Φ . In particular, if R_g is constant, g is extremal. The converse is also true if $\dim L(X) = 0$, where $L(X)$ is the maximal connected linear algebraic subgroup of $\mathrm{Aut}_0\, X$ (cf. [5]). Note also that any Kähler-Einstein metric is of constant scalar curvature.

As for the existence of Kähler metrics with R_g constant and also of extremal Kähler metrics some necessary conditions have been obtained in [15] [4](cf. Lemma 9 below) and [9]. Accordingly, there exist many examples of compact Kähler manifolds which admit no such metrics (cf. [14] and Remark 2 below). Though the above conditions become trivial in the case $L(X) = \{e\}$, Burns and de Baltolomeis [1] found examples of ruled surfaces X with a Kähler class γ such that $L(X) = \mathrm{Aut}_0 X = \{e\}$ and γ contains no extremal Kähler metric. On the other hand, as for the uniqueness, nothing seems to be known for the Kähler-Einstein case (cf. [2] [22]).

Note also that this problem of uniqueness and existence is important from the moduli point of view; in [8] we have constructed the moduli space of extremal Kähler metrics, and the uniqueness and existence correspond respectively to the injectivity and surjectivity of the natural map of this space into the corresponding moduli space of compact Kähler manifolds. It would therefore be of interest to study the problem even in a very special case. In this note we study the case of ruled manifolds over a compact Riemann surface and gather some remarks for this

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