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THE MODULAR EQUATION AND MODULAR FORMS OF WEIGHT ONE

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Dedicated to Martin Eichler

§ 1. Introduction

This is a continuation of the previous paper [8] concerning the relation between the arithmetic of imaginary quadratic fields and cusp forms of weight one on a certain congruence subgroup. Let K be an imaginary quadratic field, say $K = \boldsymbol{Q}(\sqrt{-q})$ with a prime number $q \equiv -1 \bmod 8$, and let h be the class number of K. By the classical theory of complex multiplication, the Hilbert class field L of K can be generated by any one of the class invariants over K, which is necessarily an algebraic integer, and a defining equation of which is denoted by

$$\Phi(x) = 0 .$$

The purpose of this note is to establish the following theorem concerning the arithmetic congruence relation for $\Phi(x)$:

THEOREM I. *Let p be any prime not dividing the discriminant D_Φ of $\Phi(x)$, and $\boldsymbol{F}_p$ the p-element field. Suppose that the ideal class group of K is cyclic. Then we have*

$$\#\{x \in \boldsymbol{F}_p \colon \Phi(x) = 0\} = \frac{h}{6} a(p)^2 + \frac{h}{6} a(p) - \frac{1}{2}\left(\frac{-q}{p}\right) + \frac{1}{2} ,$$

where $a(p)$ denotes the pth Fourier coefficient of a cusp form which will be defined by (1) *in Section* 2.3 *below. One notes that in case $p = 2$, we have $(-q/p) = 1$.*

§ 2. Proof of Theorem I

2.1. Let Λ be a lattice in the complex plane $\boldsymbol{C}$, and define

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