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A CRITERION FOR INTERSECTION MULTIPLICITY ONE

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Let X and Y be any pure dimensional subschemes of $\boldsymbol{P}_K^n$ over an algebraically closed field K and let $I(X)$ and $I(Y)$ be the largest homogeneous ideals in $K[x_0, \cdots, x_n]$ defining X and Y, respectively. By a pure dimensional subscheme X of $\boldsymbol{P}_K^n$ we shall always mean a closed pure dimensional subscheme without imbedded components, i.e., all primes belonging to $I(X)$ have the same dimension.

A subvariety V of $\boldsymbol{P}_K^n$ is a reduced and irreducible closed subscheme of $\boldsymbol{P}_K^n$; that is, a subvariety corresponds to a prime ideal in $K[x_0, \cdots, x_n]$.

The aim of this note is to prove a criterion of multiplicity one for isolated (i.e., irreducible) components $C \subset X \cap Y$. In case of proper components, that is, $\dim C = \dim X + \dim Y - n$, a criterion of multiplicity one is given as a statement on transversality (see, e.g., A. Weil [20], Chapter 6, Theorem 6). Using the approach to the intersection theory of Fulton and MacPherson one obtains a new proof for such a criterion (see W. Fulton [3], Propositions 7.2 and 8.2). However, in [3] we have no observations about a criterion for intersection multiplicity one for improper components.

For improper components of the intersection of subvarieties P. Samuel improves the classical result on transversality (see [12], Chap. II, § 5, no. 3). In case of the intersection theory of analytic geometry E. Selder [13] develops such a criterion for proper components of pure dimensional analytic sets. Hence our algebro-geometric proof does yield an algebraization of Selder's consideration. Moreover, our results extend Samuel's and Fulton's observations. Applying our criterion for intersection multiplicity one we see that the problem posed in [17], p. A-4 is not true in general, see our Example 8. Knowing our remark after the proof of Lemma 4 and Fulton's observation in [3], p. 174 (last period) one now expected such examples.

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