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TIGHT CLOSURE IN F-RATIONAL RINGS

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Introduction

All rings are commutative with unit and modules are unital. With one exception all rings are Noetherian. We consider only rings of positive characteristic p. Section 0 contains background, definitions and terminology.

Peskine and Szpiro showed in [PS] that when the ring R has positive characteristic the Frobenius endomorphism may be exploited with strong results. More recently, Hochster and Huneke have developed the theory of tight closure and obtained numerous results for equicharacteristic rings containing a field. If R is a Noetherian ring of characteristic p and I is an ideal of R, then $x \in R$ is in the tight closure of I, denoted I^*, if there exists an element c not in any minimal prime of R such that $cx^{p^e} \in I^{[p^e]} = (i^{p^e} : i \in I)R$ for all $e \gg 0$. Using the Frobenius endomorphism there is a definition of tight closure for finitely generated modules as well. When a module $N \subseteq M$ is equal to its tight closure then we say that N is tightly closed in M. See §0 for a more thorough exposition of tight closure. Statements which may be proved in characteristic p using tight closure arguments may then often be proved in the equicharacteristic 0 case using the techniques of Artin approximation (see [H]).

The impetus for this paper was the question: if $I \subseteq R$ is tightly closed and R/I has finite projective dimension (or more generally, finite phantom projective dimension) then is $I^{[p^e]}$ tightly closed and what other modules of finite (phantom) projective dimension have their zero submodule tightly closed? The key to the answers we have obtained are contained in Lemmas 1.2 and 1.3, in which we note that, given a chain map of finite free acyclic complexes, certain tight closure modules determined by the complexes are related.

When (R, m) is local and every ideal generated by parameters is tightly closed then R is called F-rational (see [HH3, §4]). When R is equidimensional and the homomorphic image of a Cohen-Macaulay ring then it suffices that one full system of parameters be tightly closed, in which case R is Cohen-Macaulay. In

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