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PROJECTIVE SURFACES WITH K–VERY AMPLE LINE BUNDLES OF DEGREE $\leq 4K + 4$

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Introduction

A line bundle, L, on a smooth, connected projective surface, S, is defined [7] to be k-very ample for a non-negative integer, k, if given any 0-dimensional subscheme $(Z, \mathcal{O}_Z) \subset S$ with length $(Z, \mathcal{O}_Z) \leq k + 1$, it follows that the restriction map $\Gamma(L) \to \Gamma(L \otimes \mathcal{O}_Z)$ is onto. L is 1-very ample (respectively 0-very ample) if and only if L is very ample (respectively spanned at all points by global sections). For a smooth surface, S, embedded in projective space by $|L|$ where L is very ample, L being k-very ample is equivalent to there being no k-secant $\mathbf{P}^{k-1}$ to S containing $\geq k + 1$ points of S.

In this article we study pairs (S, L), where S is a smooth, projective surface and L is a k-very ample line bundle satisfying $L \cdot L \leq 4k + 4$.

In [8] M. Beltrametti and the second author studied the question of when L being k-very ample implies that $K_S \otimes L$ is k-very ample. This question generalizes classical questions for very ample bundles, and has a nice interpretation as a question about adjunction on $S^{[k]}$, the space of 0-dimensional subschemes of length k on S (see the introduction to [8] for details).

That question breaks up naturally into the cases when $d := L \cdot L \geq 4k + 5$ and the cases when $d \leq 4k + 4$. In [8], Beltametti and the second author gave a complete answer to the question for $d \geq 4k + 5$ using their generalization, [8], of the Reider criterion for spannedness and very ampleness. This division into two parts exists in the classical case for very ample line bundles (see [18]).

In §2 and §3 we prove a number of general results for k-very ample line bundles on curves and surfaces respectively.

With these results we turn in §4 to the study of special pairs (S, L) with $d \leq 4k + 4$, mainly $\mathbf{P}^1$-bundles and k-conic bundles. The study of such special classes is required by our approach based on [8, Theorem (3.1)]. That theorem says that either (S, L) is on a list of very special pairs or $kK_S + L$ is spanned

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