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THE WEITZENBÖCK FORMULA FOR THE BACH OPERATOR

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Dedicated to Professor Tsunero Takahashi on his sixtieth birthday

§1. Introduction

(Anti-)self-dual metrics are 4-dimensional Riemannian metrics whose Weyl conformal tensor W half vanishes. The Weyl conformal tensor W of an arbitrary metric on an oriented 4-manifold has in general the self-dual part W^+ and the anti-self-dual part W^- with respect to the Hodge star operator $*$ and one says that a metric is self-dual or anti-self-dual if $W^- = 0$ or $W^+ = 0$, respectively.

Because of the conformal invariance of the defining equations $W^\pm = 0$ (anti-)self-dual metrics are, as a generalization of conformally flat metrics, an object of great interest from conformal geometry.

The notion of (anti-)self-duality of metrics depends on a choice of orientation so that a self-dual metric becomes anti-self-dual when we reverse the orientation. However, we are mainly interested in anti-self-dual metrics, unless especially mentioned.

Consider the unit sphere bundle $Z_M = U(\Omega_M^+)$ over an oriented Riemannian 4-manifold M. Then the vanishing of the self-dual part of the Weyl tensor gives an integrable condition for the almost complex structure naturally defined on Z_M. So Z_M becomes a 3-dimensional complex manifold having a smooth fibration over M with fibers $\mathbf{C}P^1$ and a fixed-point free anti-holomorphic involution, called the real structure.

The Penrose twistor theories assert that elliptic differential operators geometrically arising over an anti-self-dual 4-manifold M relate to the $\bar{\partial}$-operators on certain holomorphic vector bundles over the twistor space Z_M.

Particularly, the Kodaira-Spencer complex on $Z = Z_M$;

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