J. M. Kim
Nagoya Math. J.
Vol. 140 (1995), 101–116

UNITS AND CYCLOTOMIC UNITS IN $\mathbf{Z}_p$-EXTENSIONS

JAE MOON KIM[1]

Introduction

Let p be an odd prime and d be a positive integer prime to p such that $d \not\equiv 2 \bmod 4$. For technical reasons, we also assume that $p \nmid \varphi(d)$. For each integer $n \geq 1$, we choose a primitive nth root ζ_n of 1 so that $\zeta_m^{\frac{m}{n}} = \zeta_n$ whenever $n \mid m$. Let $K = K_0 = \mathbf{Q}(\zeta_{pd})$ and $K_\infty = \cup_{n \geq 0} K_n$ be its cyclotomic $\mathbf{Z}_p$-extension, where $K_n = \mathbf{Q}(\zeta_{p^{n+1}d})$ is the nth layer of this extension. For $n \geq 1$, we denote the Galois group $\mathrm{Gal}(K_n/K_0)$ by G_n, the unit group of the ring of integers of K_n by E_n, and the group of cyclotomic units of K_n by C_n. For the definition and basic properties of cyclotomic units such as the index theorem, we refer [6] and [7]. In this paper we examine the injectivity of the homomorphism $H^1(G_n, C_n) \to H^1(G_n, E_n)$ between the first cohomology groups induced by the inclusion $C_n \to E_n$.

In [4], it is shown that the Tate cohomology group $\hat{H}^i(G_{m,n}, C_m)$ depends on the splitting of p in $\mathbf{Q}(\zeta_d)$ where $G_{m,n} = \mathrm{Gal}(K_m/K_n)$ for $m > n$. To be more precise, let k be the decomposition field of p in $\mathbf{Q}(\zeta_d)$. Then

$$\hat{H}^i(G_{m,n}, C_m) \simeq \begin{cases} (\mathbf{Z}/p^{m-n}\mathbf{Z})^{l-1} & \text{if } i \text{ is even} \\ (\mathbf{Z}/p^{m-n}\mathbf{Z})^{l} & \text{if } i \text{ is odd,} \end{cases}$$

where

$$l = \begin{cases} [k : \mathbf{Q}] & \text{if } k \text{ is real} \\ \frac{1}{2}[k : \mathbf{Q}] & \text{otherwise.} \end{cases}$$

In particular, $H^1(G_n, C_n) \simeq (\mathbf{Z}/p^n\mathbf{Z})^l$ and by taking the direct limit under the inflation maps, we have $H^1(\Gamma, C_\infty) \simeq (\mathbf{Q}_p/\mathbf{Z}_p)^l$, where $C_\infty = \cup_{n \geq 0} C_n$ and $\Gamma = \varprojlim G_n = \mathrm{Gal}(K_\infty/K_0)$.

Received April 4, 1994.
Revised October 31, 1994.
[1] This work was supported by KOSEF, 931-0100-003-2.