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CONFORMAL IMMERSIONS OF COMPACT RIEMANN SURFACES INTO THE $2n$-SPHERE ($n \geq 2$)

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The purpose of this article is to prove the following theorem:

Let n be a positive integer larger than or equal to 2, and let S^{2n} be the unit sphere in the $2n+1$ dimensional Euclidean space. Given a compact Riemann surface, we can always find a conformal and minimal immersion of the surface into S^{2n} whose image is not lying in any $2n-1$ dimensional hyperplane.

This is a partial generalization of the result by R. L. Bryant. In this papers, he demonstrates the existence of a conformal and minimal immersion of a compact Riemann surface into S^{2n}, which is generically $1:1$, when $n = 2$ ([2]) and $n = 3$ ([1]).

We start with an idea formulated by Bryant in his paper [2], which is also fundamental for our proof. Let $\mathbf{V}$ be the set of all maximal isotropic subspaces in $\mathbf{C}^{2n+1}$ with respect to the complex symmetric bilinear form, the extension of the standard inner product on $\mathbf{R}^{2n+1}$. The set $\mathbf{V}$ is a connected compact complex manifold and has a natural projection π on the unit sphere S^{2n}, defining the twistor bundle $(\mathbf{V}, \pi, S^{2n})$, where the $SO(2n+1)$-actions on $\mathbf{V}$ and on S^{2n} are equivariant under the projection π. Beginning with E. Calabi's work ([5], [6]), the twister bundle plays an important role in the geometry of minimal surfaces, or more generally harmonic maps of surfaces, in S^{2n}. (For recent developments on twistor bundles over even dimensional Riemannian symmetric spaces and their applications, we refer to Bryant [3], Burstall-Rawnsley [4]).

There is a distribution $\mathbf{T}$ on $\mathbf{V}$ perpendicular to the fibre at each point with respect to any Riemannian metric invariant under the $SO(2n+1)$-action, which is not integrable, but is holomorphic [2]. An oriented surface immersed in S^{2n} has a complex structure canonically determined by the orientation and the first fundamental form. The basic idea of Bryant's proof [2] is that if a Riemann surface M admits an anti-holomorphic immersion φ into $\mathbf{V}$ whose image is tangent to the distribution $\mathbf{T}$ at each point on M, then $\pi{,}\varphi : M \to S^{2n}$ is a minimal and conformal

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