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SOLUTIONS OF THE SECOND AND FOURTH PAINLEVÉ EQUATIONS, I

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Abstract. A rigorous proof of the irreducibility of the second and fourth Painlevé equations is given by applying Umemura's theory on algebraic differential equations ([26], [27], [28]) to the two equations. The proof consists of two parts: to determine a necessary condition for the parameters of the existence of principal ideals invariant under the Hamiltonian vector field; to determine the principal invariant ideals for a parameter where the principal invariant ideals exist. Our method is released from complicated calculation, and applicable to the proof of the irreducibility of the third, fifth and sixth equation (e.g. [32]).

In previous papers [27] and [28], we settled the problem of the irreducibility of the first differential equation P_{I} of Painlevé. Namely we proved that no solution of the first Painlevé equation is classical. So the first Painlevé equation defines highly transcendental functions different from the classical functions. The proof depends on the condition (J) introduced in [28], which is of arithmetic nature and plays an important role in the proof of the irreducibility of the first equation P_{I}.

Our framework tells us that if an ordinary algebraic differential equation of second order satisfies the condition (J), then no transcendental solution of the differential equation is classical. So for the first equation P_{I}, the proof of the irreducibility consists of two parts: (i) To prove that the first equation satisfies the condition (J); (ii) To show that the first Painlevé equation has no algebraic solutions.

In this paper we discuss in this framework the irreducibility of the second and fourth equations $P_{\mathrm{II}}(\alpha)$, $P_{\mathrm{IV}}(\alpha,\beta)$ of Painlevé:

$$P_{\mathrm{II}}(\alpha) \qquad \frac{d^2q}{dt^2} = 2q^3 + tq + \alpha,$$

$$P_{\mathrm{IV}}(\alpha,\beta) \qquad \frac{d^2q}{dt^2} = \frac{1}{2q}\left(\frac{dq}{dt}\right)^2 + \frac{3}{2}q^3 + 4tq^2 + 2(t^2-\alpha)q + \frac{\beta}{q}.$$

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