ON THE DISCRETE SUBGROUPS AND HOMOGENEOUS SPACES OF NILPOTENT LIE GROUPS

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Recently A. Malcev[1] has shown that the homogeneous space of a connected nilpotent Lie group G is the direct product of a compact space and an Euclidean space and that the compact space of this direct decomposition is also a homogeneous space of a connected subgroup of G. Any compact homogeneous space M of a connected nilpotent Lie group is of the form $M = \widetilde{G}/D$, where $\widetilde{G}$ is a connected simply connected nilpotent group whose structure constants are rational numbers in a suitable coordinate system and D is a discrete subgroup of G.

In this paper we first determine the "situations" of discrete subgroups of a connected simply connected nilpotent group. In making use of this result we may prove the results of Malcev in a different method. Then we make some considerations on the homological properties of a compact homogeneous space and show that the cohomology groups of dimensions 1 and 2 of a nilpotent Lie algebra $\mathfrak{G}_R$ over the field R of rational numbers are isomorphic to the corresponding rational cohomology groups of a compact homogeneous space of the connected simply connected nilpotent group corresponding to the Lie algebra $\mathfrak{G}$ obtained from $\mathfrak{G}_R$ by extending the ground field R to the field of real numbers. In the above discussions Hopf-Eilenberg-MacLane's theory[2] on the relations between homology and homotopy of a space will play an important rôle.

1. Let G be a Lie group. To every element L of its Lie algebra $\mathfrak{G}$ there corresponds a one-paramenter subgroup $g(t)$ such that L is the tangent vector at the unit element to the curve $g(t)$. We shall denote this one-parameter subgroup $g(t)$ by $\exp tL$ and $\exp L$ is the point of parameter 1 on this curve. If G is a connected simply connected solvable group, then G is homeomorphic to an Euclidean space and each Lie subgroup H of G corresponding to a subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ is closed and simply connected.[3]

THEOREM 1. *Let G be a connected simply connected nilpotent group with*

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1) See Malcev [8].

2) See Hopf [5], [6], Eilenberg and MacLane [3], [4].

3) See Chevalley [1].