A THEOREM ON THE CLUSTER SETS OF PSEUDO-ANALYTIC FUNCTIONS

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1. Let D be an arbitrary connected domain and $w = f(z) = u(x,y) + iv(x,y)$, $z = x + iy$, be an interior transformation in the sense of Stoïlow in D. Denote by γ a set, in D, such that D and the derived set γ' of γ have no point in common. We suppose that

(i) $u_x,\ u_y,\ v_x,\ v_y$ exist and are continuous in $D^* = D - \gamma$;

(ii) $$J(z) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} > 0 \quad \text{at every point in } D^*;$$

(iii) the function $q(z)$ defined as the ratio of the major and minor axes of an infinitesimal ellipse with centre $f(z)$, into which an infinitesimal circle with centre at each point z of D^* is transfomed by $w = f(z)$, is bounded in D^*: $q(z) \leqq A$.

$f(z)$ is then called pseudo-meromorphic (A) in D.[1]

Next, suppose that $w = f(z)$ is pseudo-meromorphic (A) in D. Let C be the boundary of D, E be a closed set of capacity[2] zero, included in C, and z_0 be a point in E. We can associate with z_0 three cluster sets $S_{z_0}^{(D)}$, $S_{z_0}^{(C)}$ and $S_{z_0}^{*(C)}$ as follows: $S_{z_0}^{(D)}$ is the set of all values α such that $\lim_{\nu\to\infty} f(z_\nu) = \alpha$ with a sequence $\{z_\nu\}$ of points tending to z_0 inside D. $S_{z_0}^{*(C)}$ is the intersection $\bigcap_r M_r$, where M_r denotes the closure of the union $\bigcup_{\zeta'} S_{\zeta'}^{(D)}$ for all ζ' belonging to the common part of $C - E$ and $U(z_0, r)$: $|z - z_0| < r$. In the particular case when E consists of a single point z_0, we denote $S_{z_0}^{*(C)}$ by $S_{z_0}^{(C)}$ for simplicity. Obviously $S_{z_0}^{(D)}$ and $S_{z_0}^{*(C)}$ are closed sets such that $S_{z_0}^{*(C)} \subset S_{z_0}^{(D)}$ and $S_{z_0}^{(D)}$ is always non-empty while $S_{z_0}^{*(C)}$ becomes empty if and only if there exists a positive number r such that $C - E$ and $U(z_0, r)$ have no point in common.

In the particular case where $w = f(z)$ is single-valued meromorphic in D, the following theorems concerning the cluster sets $S_{z_0}^{(D)}$, $S_{z_0}^{(C)}$ and $S_{z_0}^{*(C)}$ are known:

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[1] For the definition of pseudo-meromorphic functions, Cf. S. Kakutani: Applications to the theory of pseudo-regular functions to the type-problem of Riemann surfaces, Jap. Journ. of Math. Vol. **13** (1937), pp. 375-392. R. Nevanlinna: Eindeutige analytische Funktionen, Berlin, 1936, p. 343.

[2] "Capacity" means logarithmic capacity in this note.