ON MEAN DISTORTION FOR ANALYTIC FUNCTIONS WITH POSITIVE REAL PART IN A CIRCLE

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Dedicated to Professor K. NOSHIRO on his sixtieth birthday

1. Introduction

Let $\mathfrak{R}$ be the class of analytic functions $\Phi(z)$ which are regular and of positive real part in the unit circle $|z|<1$ and normalized by $\Phi(0)=1$. Several distortion theorems have been obtained on various functionals in this class. In a previous paper [4] we have dealt with mean distortion which generalizes a classical theorem of Rogosinski [6].

On the other hand, it is well known that the fundamental operations, integration and differentiation, in ordinary calculus can be analytically interpolated to those of any real order. It has been shown in [5] that such fractional calculus can be used in order to generalize the classical results on angular derivative of Julia [3] and Wolff [7] or Carathéodory [1].

In the present paper we shall show that the notion of fractional calculus is also useful in dealing with distortion theorems in the class $\mathfrak{R}$. Though the results which will be derived below are essentially involved in general theorems obtained in [4], they may be regarded as an interpolating generalization of some illustrating theorems given there. At any rate it will be of some interest to point out concrete cases where the estimates in distortion inequalities are expressed in terms of integrals of elementary functions.

2. Preliminaries

Since the unit circle is a convex domain, the fractional integration and differentiation can be defined uniquely for any analytic function regular in the domain as explained in [5] with respect to any reference point in the domain. We suppose here that the reference point lies always at the origin.

Let now q be any positive real number and $\mathscr{D}^{-q}$ denote the integration of

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